Proof. This is immediate, since $\mu_T(x)$ is the largest invariant factor of $T$, and $\chi_T(x)$ is the product of all of the invariant factors.

Let us make a few more remarks about the relation between the minimal polynomial and the characteristic polynomial. All of the invariant factors of $T$ divide the minimal polynomial $\mu_T(x)$, and $\chi_T(x)$ is the product of all of the invariant factors. It follows that $\chi_T(x)$ and $\mu_T(x)$ have the same irreducible factors, but with possibly different multiplicities. Since $\lambda \in K$ is a root of a polynomial exactly when $x - \lambda$ is an irreducible factor, we also have that $\chi_T(x)$ and $\mu_T(x)$ have the same roots, but with possibly different multiplicities. Finally, the characteristic polynomial and the minimal polynomial coincide precisely if $V$ is a cyclic $K[x]$-module; i.e., the rational canonical form of $T$ has only one block.

Of course, statements analogous to Corollary M.6.13, and of these remarks, hold for a matrix $A \in \text{Mat}_n(K)$ in place of the linear transformation $T$.

The roots of the characteristic polynomial (or of the minimal polynomial) of $T \in \text{End}_K(V)$ have an important characterization.

**Definition M.6.14.** We say that an nonzero vector $v \in V$ is an eigenvector of $T$ with eigenvalue $\lambda$, if $Tv = \lambda v$. Likewise, we say that a nonzero vector $v \in K^n$ is an eigenvector of $A \in \text{Mat}_n(K)$ with eigenvalue $\lambda$ if $Av = \lambda v$.

The words “eigenvector” and “eigenvalue” are half-translated German words. The German Eigenvektor and Eigenwert mean “characteristic vector” and “characteristic value.”

**Proposition M.6.15.** Let $T \in \text{End}_K(V)$. An element $\lambda \in K$ is a root of $\chi_T(x)$ if, and only if, $T$ has an eigenvector in $V$ with eigenvalue $\lambda$.

Proof. Exercise M.6.7

**Exercises M.6**

**M.6.1.** Let $h(x) \in K[x]$ be a polynomial of one variable. Show that there is a polynomial $g(x, y) \in K[x, y]$ such that $h(x) - h(y) = (x - y)g(x, y)$.

**M.6.2.** Set $w_j = (x - T)f_j = xf_j - \sum_i a_{i,j}f_i$. Show that $\{w_1, \ldots, w_n\}$ is linearly independent over $K[x]$. 
M.6.3. Verify the following assertions made in the text regarding the computation of the rational canonical form of $T$. Suppose that $F$ is a free $K[x]$ module, $\Phi : F \rightarrow V$ is a surjective $K[x]$–module homomorphism, $(y_1, \ldots, y_{n-s}, z_1, \ldots, z_s)$ is a basis of $F$, and
$$(y_1, \ldots, y_{n-s}, a_1(x)z_1, \ldots, a_s(x)z_s)$$
is a basis of ker$(\Phi)$. Set $v_j = \Phi(z_j)$ for $1 \leq j \leq s$, and

$V_j = K[x]v_j = \span\{p(T)v_j : p(x) \in K[x]\}$.

(a) Show that $V = V_1 \oplus \cdots \oplus V_s$.

(b) Let $\delta_j$ be the degree of $a_j(x)$. Show that $\begin{pmatrix} v_j, T v_j, \ldots, T^{\delta_j-1} v_j \end{pmatrix}$ is a basis of $V_j$, and that the matrix of $T|_{V_j}$ with respect to this basis is the companion matrix of $a_j(x)$.

M.6.4. Let $A = \begin{bmatrix} 7 & 4 & 5 & 1 \\ -15 & -10 & -15 & -3 \\ 0 & 0 & 5 & 0 \\ 56 & 52 & 51 & 15 \end{bmatrix}$. Find the rational canonical form of $A$ and find an invertible matrix $S$ such that $S^{-1}AS$ is in rational canonical form.

M.6.5. Show that $\chi_A$ is a similarity invariant of matrices. Conclude that for $T \in \operatorname{End}_K(V)$, $\chi_T$ is well defined, and is a similarity invariant for linear transformations.

M.6.6. Since $\chi_A(x)$ is a similarity invariant, so are all of its coefficients. Show that the coefficient of $x^{n-1}$ is the negative of the trace $\operatorname{tr}(A)$, namely the sum of the matrix entries on the main diagonal of $A$. Conclude that the trace is a similarity invariant.

M.6.7. Show that $\lambda$ is a root of $\chi_T(x)$ if, and only if, $T$ has an eigenvector in $V$ with eigenvalue $\lambda$. Show that $v$ is an eigenvector of $T$ for some eigenvalue if, and only if, the one dimensional subspace $Kv \subseteq V$ is invariant under $T$.

The next four exercises give an alternative proof of the Cayley-Hamilton theorem. Let $T \in \operatorname{End}_K(V)$, where $V$ is $n$–dimensional. Assume that the field $K$ contains all roots of $\chi_T(x)$; that is, $\chi_T(x)$ factors into linear factors in $K[x]$.

M.6.8. Let $V_0 \subseteq V$ be any invariant subspace for $T$. Show that there is a linear operator $\overline{T}$ on $V/V_0$ defined by
$$\overline{T}(v + V_0) = T(v) + V_0$$
for all $v \in V$. Suppose that $(v_1, \ldots, v_k)$ is an ordered basis of $V_0$, and that
$$(v_{k+1} + V_0, \ldots, v_n + V_0)$$
is an ordered basis of $V/V_0$. Suppose, moreover, that the matrix of $T|_{V_0}$ with respect to $(v_1, \ldots, v_k)$ is $A_1$ and the matrix of $\overline{T}$ with respect to $(v_{k+1} + V_0, \ldots, v_n + V_0)$ is $A_2$. Show that $(v_1, \ldots, v_k, v_{k+1}, \ldots, v_n)$ is an ordered basis of $V$ and that the matrix of $T$ with respect to this basis has the form

$$\begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix},$$

where $B$ is some $k$–by–$(n - k)$ matrix.

**M.6.9.** Use the previous two exercises, and induction on $n$ to conclude that $V$ has some basis with respect to which the matrix of $T$ is upper triangular; that means that all the entries below the main diagonal of the matrix are zero.

**M.6.10.** Suppose that $A_0$ is the upper triangular matrix of $T$ with respect to some basis of $V$. Denote the diagonal entries of $A_0$ by $(\lambda_1, \ldots, \lambda_n)$; this sequence may have repetitions. Show that $T - \lambda_k$ maps $V_k$ into $V_{k-1}$ for all $k, 1 \leq k \leq n$. Show by induction that

$$(T - \lambda_k)(T - \lambda_{k+1}) \cdots (T - \lambda_n)$$

maps $V$ into $V_{k-1}$ for all $k, 1 \leq k \leq n$. Note in particular that

$$(T - \lambda_1) \cdots (T - \lambda_n) = 0.$$

Using the previous exercise, conclude that $\chi_T(T) = 0$, the characteristic polynomial of $T$, evaluated at $T$, gives the zero transformation.

**Remark M.6.16.** The previous four exercises show that $\chi_T(T) = 0$, under the assumption that all roots of the characteristic polynomial lie in $K$. This restriction can be removed, as follows. First, the assertion $\chi_T(T) = 0$ for $T \in \text{End}_K(V)$ is equivalent to the assertion that $\chi_A(A) = 0$ for $A \in \text{Mat}_n(K)$. Let $K$ be any field, and let $A \in \text{Mat}_n(K)$. If $F$ is any field with $F \supseteq K$ then $A$ can be considered as an element of $\text{Mat}_n(F)$. The characteristic polynomial of $A$ is the same whether $A$ is regarded as a matrix with entries in $K$ or as a matrix with entries in $F$. Moreover, $\chi_A(A)$ is the same matrix, whether $A$ is regarded as a matrix with entries in $K$ or as a matrix with entries in $F$.

As is explained in Section 8.2, there exists a field $F \supseteq K$ such that all roots of $\chi_A(x)$ lie in $F$. It follows that $\chi_A(A) = 0$. 
