The periods $p_j^{n_{i,j}}$ of the direct summands in the decomposition described in Theorem M.5.13 are called the *elementary divisors* of M. They are determined up to multiplication by units.

Example M.5.14. Let

$$f(x) = x^5 - 9x^4 + 32x^3 - 56x^2 + 48x - 16$$

and

$$g(x) = x^{10} - 6x^9 + 16x^8 - 30x^7 + 46x^6 - 54x^5 + 52x^4 - 42x^3 + 25x^2 - 12x + 4.$$

Their irreducible factorizations in $\mathbb{Q}[x]$ are

$$f(x) = (x-2)^4(x-1)$$

and

$$g(x) = (x-2)^2(x-1)^2\left(x^2+1\right)^3.$$

Let M denote the $Q[x]$–module $M = \mathbb{Q}[x]/(f) \oplus \mathbb{Q}[x]/(g)$. Then

$$\begin{aligned} M &\cong \mathbb{Q}[x]/((x-2)^4) \oplus \mathbb{Q}[x]/((x-1)) \\ &\quad \oplus \mathbb{Q}[x]/((x-2)^2) \oplus \mathbb{Q}[x]/((x-1)^2) \oplus \mathbb{Q}[x]/((x^2+1)^3) \end{aligned}$$

The elementary divisors of M are $(x-2)^4, (x-2)^2, (x-1)^2, (x-1)$, and $(x^2+1)^3$. Regrouping the direct summands gives:

$$\begin{aligned} M &\cong \left(\mathbb{Q}[x]/((x-2)^4) \oplus \mathbb{Q}[x]/((x-1)^2) \oplus \mathbb{Q}[x]/((x^2+1)^3)\right) \\ &\oplus \left(\mathbb{Q}[x]/((x-2)^2) \oplus \mathbb{Q}[x]/((x-1))\right) \\ &\cong \mathbb{Q}[x]/((x-2)^4(x-1)^2\left(x^2+1\right)^2) \oplus \mathbb{Q}[x]/((x-2)^2(x-1)). \end{aligned}$$

The invariant factors of M are $(x-2)^4(x-1)^2\left(x^2+1\right)^3$ and $(x-2)^2(x-1)$.

Exercises M.5

M.5.1. Let R be an integral domain, M an R–module and S a subset of R. Show that $\mathrm{ann}(S)$ is an ideal of R and $\mathrm{ann}(S) = \mathrm{ann}(RS)$.

M.5.2. Let M be a module over an integral domain R. Show that M/M_{tor} is torsion free

M.5.3. Let M be a module over an integral domain R. Suppose that $M = A \oplus B$, where A is a torsion submodule and B is free. Show that $A = M_{\mathrm{tor}}$.

M.5.4. Let R be an integral domain. Let B be a maximal linearly independent subset of an R–module M. Show that RB is free and that M/RB is a torsion module.

M.5.5. Let R be an integral domain with a non–principal ideal J. Show that J is torsion free as an R–module, that any two distinct elements of J are linearly dependent over R, and that J is a not a free R–module.

M.5.6. Show that $M = \mathbb{Q}/\mathbb{Z}$ is a torsion $\mathbb{Z}$–module, that M is not finitely generated, and that $\text{ann}(M) = \{0\}$.

M.5.7. Let R be a principal ideal domain. The purpose of this exercise is to give another proof of the uniqueness of the invariant factor decomposition for finitely generated torsion R–modules.

Let p be an irreducible of R.

(a) Let a be a nonzero, nonunit element of R and consider $M = R/(a)$. Show that for $k \geq 1$, $p^{k-1}M/p^k M \cong R/(p)$ if p^k divides a and $p^{k-1}M/p^k M = \{0\}$ otherwise.

(b) Let M be a finitely generated torsion R–module, with a direct sum decomposition

$$M = A_1 \oplus A_2 \oplus \cdots \oplus A_s,$$

where

- for $i \geq 1$, $A_i \cong R/(a_i)$, and
- the ring elements a_i are nonzero and noninvertible, and a_i divides a_j for $i \geq j$;

Show that for $k \geq 1$, $p^{k-1}M/p^k M \cong (R/(p))^{m_k(p)}$, where $m_k(p)$ is the number of a_i that are divisible by p^k. Conclude that the numers $m_k(p)$ depend only on M and not on the choice of the direct sum decomposition $M = A_1 \oplus A_2 \oplus \cdots \oplus A_s$.

(c) Show that the numbers $m_k(p)$, as p and k vary, determine s and also determine the ring elements a_i up to associates. Conclude that the invariant factor decomposition is unique.

M.5.8. Let M be a finitely generated torsion module over a PID R. Let m be a period of M with irreducible factorization $m = p_1^{m_1} \cdots p_s^{m_s}$. Show that for each i and for all $x \in M[p_i]$, $p_i^{m_i} x = 0$.

M.6. Rational canonical form

In this section we apply the theory of finitely generated modules of a principal ideal domain to study the structure of a linear transformation of a finite dimensional vector space.

If T is a linear transformation of a finite dimensional vector space V over a field K, then V has a $K[x]$–module structure determined by