Practice Exam 4: Topology

June 2, 2005

Directions: This is a three hour closed book exam. There are two parts of the exam. Work four questions from part 1 and four questions from part 2.

1 Part 1: General Topology

1. Let \( Y \) be Hausdorff and suppose \( f, g : X \rightarrow Y \) are continuous. Prove that the set \( \{ x \mid f(x) = g(x) \} \) is a closed subset of \( X \).

2. Prove that the continuous image of a compact set is compact. Prove that the continuous image of a connected set is connected.

3. Prove that any space \( X \) is locally connected if and only if the connected components of each open set are open.

4. Prove that any regular second countable space is normal.

5. Prove that if \( A, B \) are closed subsets of \( X \) with \( A \cup B = X \), and \( f|_A \) and \( g|_B \) are continuous, and \( f \) and \( g \) agree on \( A \cap B \), then \( f \cup g : X \rightarrow Y \) is continuous.

6. Prove that a metric space \( X \) is compact if and only if it is sequentially compact.

2 Part 2: Smooth Manifolds

1. Let \( SL_2(\mathbb{R}) \) denote the Lie group of two by two real matrices of determinant 1. Write out formulas for the left invariant vector fields coming
from the tangent vectors at the identity \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \). Then compute their Lie Bracket.

2. Let \( V \) be a vector space and let \( Alt : \mathcal{T}(V) \to \mathcal{A}(V) \) be the standard map that takes tensors to alternating tensors. Prove that the kernel of \( Alt \) is an ideal. That is, if \( Alt(T) = 0 \) then for any tensor \( S \),

\[
Alt(T \otimes S) = 0.
\]

3. Prove that any \( C^\infty \)-compatible atlas on a manifold \( M \) is contained in a maximal compatible atlas.

4. Use stereographic projection from the north pole to define a coordinate patch on \( S^2 \). Compute the coefficients of the metric tensor induced from the inclusion in \( \mathbb{R}^3 \) in these coordinates.

5. Prove that the inclusion map \( i : M \to N \) of a smooth submanifold is an immersion.

6. Prove that if \( M \) is a smooth submanifold of \( N \) then \( f : M \to \mathbb{R} \) is smooth if and only if at each \( P \in M \) there is \( U \) open in \( N \) containing \( P \) and a smooth function \( \overline{f} : U \to \mathbb{R} \) so that \( \overline{f}|_{M \cap U} = f|_{M \cap U} \).