Notes on infinite sets

Since we cannot ever finish counting an infinite set, we need a different approach to thinking about "how many elements" such a set contains. In the early 1900s, Georg Cantor presented an idea that has clarified thinking about this issue and had enormous impact on opening new pursuits in logic.

Two (finite or) infinite sets A and B are said to have the **same cardinality** (equipotent) if there is a one-to-one, onto function (i.e., a bijection) $f: A \to B$.

Since it is impossible to ascribe an ordinary number to the size of an infinite set, we don't try. Instead, the comparative idea above is used. If we can "line the elements up" in exactly corresponding pairs, we say the two sets have the *same cardinality* (i.e., size).

So, for instance, the set of natural numbers $N = \{0, 1, 2, \dots\}$ has the same cardinality as the collection of all strings over the alphabet $\{0,1\}$. We can line them up as follows:

N	0	1	2	3	4	5	6	7	8	9	…
$\{0,1\}^*$	ε	0	1	00	01	10	11	000	001	010	…

It should be intuitively clear that we have a 1-1, onto correspondence. However, especially with the potentially elusive idea of infinity, we need to be able to follow this up with a precise function definition that permits a convincing demonstration of the desired properties of the correspondence. This is often not so easy.

In this case, for $f: N \to \{0,1\}^*$, we can say for each $n \geq 0$ and $2^n - 1 \leq m \leq 2^{n+1} - 2$, define $f(m)$ as the n-bit binary expansion of $m - (2^n - 1)$. This provides a function $f: N \to \{0,1\}^*$ that we can show is 1-1 and onto. Since each integer m falls uniquely between two successive powers of 2, given m, there is only one corresponding power of 2 in this definition that "brackets" m. And each m clearly has a unique n-bit binary expansion, so f is 1-1. To see that f is onto, pick any string $x \in \{0,1\}^*$. Then the length of x determines the number n, and the number m for which the n-bit binary expansion of $m - (2^n - 1)$ is x has $f(m) = x$, so f is onto.

An *infinite cardinality* then refers to the collection of all sets "with the same number of elements" as a given infinite set — e.g., N and $\{0,1\}^*$. In the case of finite sets, this comparative idea agrees perfectly with the counting idea. But for infinite sets, we see that a set can have the same cardinality as one of its proper subsets! Since our intuition about set size from the finite sets may be misleading, our conclusions about the "size" of infinite sets must be carefully reasoned.

Note that $N = \{0, 1, 2, \dots\}$ and $N' = \{1, 2, 3, \dots\}$ have the same cardinality. For a bijection $f: N \to N'$, take $f(n) = n+1$ (can you show 1-1 & onto?). Thus the proper subset with one item deleted has the same cardinality as the original set — this *never* happens with finite sets! But it's much more dramatic than that. For instance, $N = \{0, 1, 2, \dots\}$ and $\{0, 2, 4, \dots\} = Ev$ have the same cardinality. For a bijection $f: N \to Ev$ take $f(n) = 2*n$

(likewise 1-1 & onto). So even if we delete "half the items", we still have just as many as we started with!

We can formalize the idea of larger infinities. We say that S_1 has a **larger cardinality** than the set S_2 if S_2 has the same cardinality as a proper subset of S_1, but S_2 does not have the same cardinality as S_1.

Initial ideas about finding an infinity larger than N are likely to be unsuccessful. For instance, consider the collection of all positive and negative integers, Z = { … -2, -1, 0, 1, 2, …}. This is a superset of N, but it still has the same cardinality. For instance,

N	0	1	2	3	4	5	6	7	8	9	…
Z	0	-1	1	-2	2	-3	3	-4	4	-5	…

What is the general rule for this correspondence?

As an apparently even larger set, we could consider N×N (i.e., all pairs of natural numbers). For finite sets, we earlier noted that #(A×A) = #A ∗ #A and in general this product is much larger than #A. However, N and N×N have the same cardinality! To justify this assertion, we need to identify a 1-1, onto correspondence. This is considerably more challenging for these sets. This was outlined in class and in our text.

Infinite sets that have the same cardinality as N = {0, 1, 2, … } are called **countably infinite**. A set that has a larger cardinality than this is called **uncountably infinite**.

Assertion: there are uncountably infinite sets.
Proof by contradiction:
Let [0,1] denote the interval of all real numbers x, 0≤x≤1 — this set is uncountably infinite. First of all, note that there is an injection f: N → [0,1] defined by f(n) = n/(n+1). This maps N to a proper subset of [0,1], so [0,1] has cardinality at least as large as N.

To show that the cardinality of [0,1] is actually larger than N, we proceed by contradiction. Suppose that [0,1] *is* countably infinite. Suppose that g :N → [0,1] is a bijection, and therefore {g(n) | n∈N} is the set of *all* real number in the interval [0,1]. We show that there must be at least one real number missing in this enumeration providing a contradiction to the existence of g, and hence there is no bijection. Specifically, we construct the decimal expansion of a number α that is missing from {g(n) | n∈N}. Take the decimal expansion of α to be $.d_1d_2d_3$ …, where d_n = 5 if the nth digit of g(n) is 6, and d_n = 6 otherwise. Then for each n, the nth digit of α differs from the nth digit of g(n). Therefore α differs from every number in the enumeration in at least one digit and so $\alpha \notin$ {g(n) | n∈N}, a contradiction since this set was assumed to be *all* such numbers.