Notes on Euler Paths

Definition: an **Euler path** in a (multi) graph or digraph is a path in which each edge appears exactly once.

Theorem: an undirected multigraph G with no self-loops and no isolated nodes has an Euler cycle if and only if:
(1) G is connected, and
(2) all node degrees are even.

Theorem: an undirected multigraph G with no self-loops and no isolated nodes has a non-cyclic Euler path if and only if:
(1) G is connected, and
(2) exactly two node degrees are odd and all others are even.

The town of Königsberg straddles a river (sides A and D) and includes two islands (B and C). The various sectors of the town are connected by seven bridges in the manner indicated in Figure 1. A long standing question among the residents was whether or not there is a point from which one can begin a walk, cross each bridge precisely once, and return to the place you began. What is the answer to this question, and why?

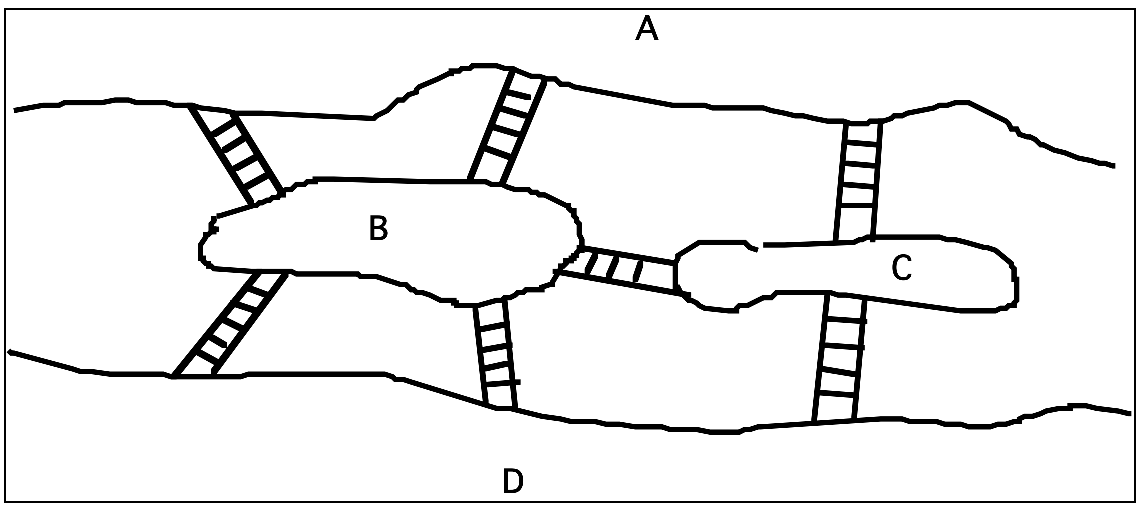

Figure 1.

The corresponding graph (multigraph actually) has four nodes, one for each of the regions A, B, C, and D. There is one edge connecting each of these nodes for each bridge that adjoins the corresponding region. Draw this graph and count the degrees to answer this question.

Another similar question is: can you start at one of the nodes in Figure 2 and draw the figure without lifting your pencil or retracing lines? Justify your answer. Since we are not required to stop and start at the same point, this is transformed into an Euler pat h problem. The technique we used to prove the Euler path/cycle theorems, tells us if this problem can be solved, and if so, how to go about constructing a solution. Give this a try.

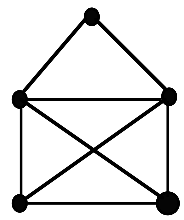

Figure 2.