Home work 2 sample solution

22C:034 Spring 2004

Q1:

$$RR(n) = \frac{n \times (n-1)}{2}. \tag{1}$$

Solution Let $P(n)$ be the property of n that (1) holds. We now prove by mathematical induction that $p(n)$ is true for all $n \geq 2$.

Inductive base $p(2)$ is true because in a round-robin tournament with $n = 2$ the number of matches, RR(2) is 1. In this case, $\frac{n\times(n-1)}{2} = 1$.

Inductive hypothesis The inductive hypothesis is given by (1) with n fixed.

Inductive step In a round-robin tournament with n+1 players, $s_1, s_2, \cdots, s_n, s_{n+1}$, the number of matches among the n players $s_1, s_2, \cdots, s_n$ are RR(n), and the number of matches with player s_{n+1} are n, because s_{n+1} needs to play a match with each of $s_1, s_2, \cdots, s_n$.

So the total number of matches for in a round-robin tournament with n+1 players are RR(n+1), RR(n+1)= RR(n)+n .

One has

RR(n+1) left side of (1) with $n := n + 1$

= RR(n) + n from above

$= \frac{n\times(n-1)}{2} + n$ Inductive hypothesis

$= \frac{(n+1)\times((n+1)-1)}{2}$ right side of (1) with $n := n + 1$

So $p(n + 1)$ holds.

Conclusion The inductive base and the inductive step imply that (1) is valid for all $n \geq 2$.

Q2:

Solution We prove by induction on the structure of fpe e that

$P(e) \equiv$ if e obtains no negations then the number of symbols in e are odd.

holds for all fpe e .

Inductive base e is an atomic expression, i.e., a single propositional variable or a single propositional constant. In this case the number of symbols in e is 1, which is odd.

Inductive Hypothesis Assume that, for any subexpression A of e , if A contains no negations, then the number of symbols in A are odd.

Inductive step

We only need to consider the cases of $e \equiv (A \wedge B)$, $e \equiv (A \vee B)$, $e \equiv (A \Rightarrow B)$,and $e \equiv (A \Leftrightarrow B)$.

Case 1: $e \equiv (A \wedge B)$

If e contains no negations, then subexpression A and B contains no negations. By inductive hypothesis, A and B both have odd number of symbols. The logical connective $\wedge$ is one single symbol. Odd number + odd number + 1 = odd number. So the number of symbols in e are odd.

The proof for the cases $e \equiv (A \vee B)$, $e \equiv (A \Rightarrow B)$,and $e \equiv (A \Leftrightarrow B)$ are similar.

Conclusion One concludes that $P(e)$ holds for all fpe e .

Q3:

$$\sum_{k=0}^{n} k \cdot (k+1) = \frac{2n^3 + 6n^2 + 4n}{6} \tag{2}$$

Solution Let $P(n)$ be the property of n that (2) holds. We now prove by mathematical induction that $p(n)$ is true for all natural numbers n .

Inductive base $p(0)$ is true because (2) holds for $n = 0$. In this case, $\sum_{k=0}^{0} k \cdot (k+1) = 0$, and

Inductive hypothesis The inductive hypothesis is given by (2) with n fixed.

Inductive step One has
$\sum_{k=0}^{n+1} k \cdot (k+1)$ left side of (2) with $n := n+1$
$= \sum_{k=0}^{n} k \cdot (k+1) + (n+1) \cdot (n+2)$ by the definition
$= \frac{2n^3+6n^2+4n}{6} + (n+1) \cdot (n+2)$ Inductive hypothesis
$= \frac{n \cdot (n+1) \cdot (n+2)}{3} + (n+1) \cdot (n+2)$
$= \frac{n \cdot (n+1) \cdot (n+2) + 3 \cdot (n+1) \cdot (n+2)}{3}$
$= \frac{(n+1) \cdot (n+2) \cdot (n+3)}{3}$
$= \frac{2(n+1)^3+6(n+1)^2+4(n+1)}{6}$ right side of (2) with $n := n+1$
So $p(n+1)$ holds.

Conclusion The inductive base and the inductive step imply that (2) is valid for all n .

Q4:
(One has
$a_1.(a_2.(....(a_n.[]) \cdots)) = [a_1, a_2, \cdots, a_n]$ (a)
cat([].y) = y (b)
cat(a.x, y) = a.cat(x,y) (c))

$$cat([a_1, a_2, \cdots, a_n], [b_1, b_2, \cdots, b_m]) = [a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_m] \tag{3}$$

Proof Let $P(n)$ be the proposition that (3) is true for all $n \geq 0$,assuming m is fixed and $m \geq 0$. One has the following:

Base for induction For n=0, (3) is true. In this case, one has

$cat([], [b_1, b_2, \cdots, b_m]) = [b_1, b_2, \cdots, b_m]$ by (b).

Inductive hypothesis The inductive hypothesis is given by (3) with n fixed.

Inductive step One has
$cat([a_1, a_2, \cdots, a_n, a_{n+1}], [b_1, b_2, \cdots, b_m])$ Left side of (3)
$= cat(a_1.[a_2, \cdots, a_n, a_{n+1}], [b_1, b_2, \cdots, b_m])$ By (a)
$= a_1.cat([a_2, \cdots, a_n, a_{n+1}], [b_1, b_2, \cdots, b_m])$ By (c)
$= a_1.[a_2, \cdots, a_n, a_{n+1}, b_1, b_2, \cdots, b_m]$ By induction hypothesis
$= [a_1, a_2, \cdots, a_n, a_{n+1}, b_1, b_2, \cdots, b_m]$ By (a) Right side of (3)
So $p(n+1)$ holds.

Conclusion The inductive base and the inductive step imply that (3) is valid for all n .

Q5:
Example $A = a, b, c$, $B = b$, $C = c$. One has $A \cup B = a, b, c, and A \cup C = a, b, c, but B \neq C$.

Q6:
Prove

$$A \cap (\sim A \cup B) = A \cap B \tag{4}$$

Proof
$A \cap (\sim A \cup B)$ left side of (4)
$= (A \cap \sim A) \cup (A \cap B)$ by Distributive Laws
$= \phi \cup (A \cap B)$ By Exclusion Law
$= (A \cap B) \cup \phi$ By Commutative laws
$= A \cap B$ By Identity Laws
This completes the proof of (4) .