Homework IX

1. [15 points]
Consider the recursive program over all integers $Z$

$$P: f(x,y) = \text{if } x > 10 \text{ then } x-1 \text{ else } f(x+2, f(x,y+1)).$$

Determine the least fixed point of the functional $P$ (i.e., the recursive function defined using lazy evaluation) and justify your answer.

2. [20 points]
(a) Show that if $X$ is a pointed cpo, $C$ is a chain of $X$, and $f: X \to X$ is a monotone function whose domain includes $C$, then $f(C) = \{f(x) \mid x \in C\}$ is also a chain.
(b) Suppose we define a function $mp: \text{Nat} \to \text{Nat}$, where $\text{Nat}$ and $\text{Nat} \to \text{Nat}$ are the pointed cpos from our text, by

$$mp(\Diamond\text{Nat}, \Diamond\text{Nat}) = \Diamond\text{Nat},$$

$$mp(0, \Diamond\text{Nat}) = 0,$$

$$mp(\Diamond\text{Nat}, 0) = 0,$$

$$mp(m,n) = m \cdot n \text{ for } m,n \in \{0, 1, 2, \ldots \}.$$  

Determine whether or not $mp$ is monotone and continuous (justify your answer).

3. [25 points]
Let $\square$ be the Wren program fragment

$$\text{if } A = B \text{ or } A = -B$$

$$\text{then if } A > B \text{ then } C := A \text{ else } C := B - A + 1 \text{ end if}$$

$$\text{else if } A \cdot B > 0 \text{ then } C := A \cdot B \text{ else } C := 1 - A \cdot B \text{ end if}$$

$$\text{end if}$$

Assuming a store, $sto$, where $sto(A)$ and $sto(B)$ denote (positive, negative, or zero) integer values, use the denotational semantics of Wren to argue that if execute $\square$ sto = sto', then sto'(C) = int(c) with $c > 0$. 