Propositional Logic

Definition: the (well formed) formulas (wffs) over a set of variables $V$ consist of:
(i) $T$, $F$, and each $X \in V$,
(ii) for each pair of wffs $\square$ and $\square$
  * $(\square \square)$,
  * $(\square \square)$,
  * $(\square \square)$,
  * $(\square \square)$,
  * $(\square \square)$,

To avoid excessive parentheses in wffs we adopt the following precedence and take $\square$ to be right-associative

<table>
<thead>
<tr>
<th>highest precedence</th>
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</thead>
<tbody>
<tr>
<td>$\square$</td>
</tr>
<tr>
<td>$\square$, $\iff$</td>
</tr>
<tr>
<td>lowest precedence</td>
</tr>
</tbody>
</table>

The definitions of the logical connectives are given by their “truth-tables”

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$\square P$</th>
<th>$P \square Q$</th>
<th>$P$</th>
<th>$Q$</th>
<th>$P \iff Q$</th>
</tr>
</thead>
<tbody>
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Definition: an assignment to a set $V$ of variables is a function $\square : V \to \{T,F\}$. Each assignment is inductively extended to apply to wffs. For wffs $\square$ and $\square$

* $\square (\square) = \square (\square)$,
* $\square (\square \square) = \square (\square) \iff \square (\square)$,
* $\square (\square \square) = \square (\square) \square (\square)$,
* $\square (\square \square) = \square (\square) \square (\square)$,
* $\square (\square \iff \square) = \square (\square) \iff \square (\square)$, and
* $\square (T) = T$, $\square (F) = F$,

Definition: wffs $\square$ and $\square$ are logically equivalent, $\square \equiv \square$, if $\square (\square) = \square (\square)$ for each assignment $\square$.

Definition: Let $\square$ and $\square$ be wffs. Then $\square$ is satisfiable if there is an assignment $\square$ so that $\square (\square) = T$; an unsatisfiable wff is also called a contradiction. If $\square (\square) = T$ for every assignment $\square$, then $\square$ is a tautology.

Definition: a set of wffs $\mathcal{S}$ logically implies a wff $\square$, $\mathcal{S} \models \square$, provided that for each assignment $\square$ such that $\mathcal{S} (\square) = T$ for each $\square \in \mathcal{S}$, $\square (\square) = T$ (if $\mathcal{S} = \emptyset$, write $\models \square$ and $\square$ is a tautology).
Definition: a **proof system** consists of the following constituents

- a subset of wffs called **axioms** — we expect there is a decision procedure to effectively determine whether or not a wff is an axiom.
- a finite collection \{R_1, \ldots, R_n\} of **inference rules**, where each rule \(R_i\) allows us to decide for wffs \(\Box_1, \ldots, \Box_{m_i}\), whether or not \(\Box\) is a *direct consequence* of \(\Box_1, \ldots, \Box_{m_i}\), written \(\Box_1 \ldots \Box_{m_i} \vdash \Box\).
- **proofs** which are sequences \(\Box_1, \ldots, \Box_k\) of wffs so that each \(i (1 \leq i \leq k)\), either \(\Box_i\) is an axiom or \(\Box_i\) is a direct consequence of some preceding wffs in the sequence; the last wff of a proof is called a **theorem**.

Two important rules of inference are:

- **modus ponens** — for any wffs \(\Box\) and \(\Box \rightarrow \Box\)
- **resolution** — for any wffs \(\Box\), \(\Box\), and \(\Box \rightarrow \Box\).

Definition: in a proof system with axioms \(A\), a wff \(\Box\) is a **consequence of a set of wffs** \(\Box\) if it is a theorem in the proof system with axioms \(A \cup \Box\). The elements of \(\Box\) are called **hypotheses** or **premises** and we write \(\Box \vdash \Box\); if \(\Box = \emptyset\), write \(\vdash \Box\) (i.e., with no premises, the consequences are just the theorems).

Definition: a rule of inference \(\Box_1 \ldots \Box_{m_i} \vdash \Box\) is **sound** provided that whenever each \(\Box_j (1 \leq j \leq m_i)\) is a tautology, \(\Box\) is also a tautology. A proof system is **sound** if each theorem is a tautology. A proof system is **complete** if each tautology is a theorem.