

Isomorphism

One of the fundamental ideas in algebra is also significant in algebraic specification — it is called *isomorphism*. It is a technical term for what we often refer to as "renaming". It consists of a correspondence between systems that serves to identify them as "essentially the same". Here is its definition.

Definition: Given two algebraic systems, A_1 with sets S_1, \dots, S_n ; and operations f_1, \dots, f_n and A_2 with sets T_1, \dots, T_n , and operations g_1, \dots, g_n , an **isomorphism** consists of a one-to-one, onto correspondence (bijection) of the sets and operations $\alpha: S_k \rightarrow T_k$ and $\alpha: f_k \rightarrow g_k$ ($1 \leq k \leq n$), where f_k and g_k have *similar* signatures (i.e., domain and range sets are in correspondence), and for all x_i and f_k

$$\alpha(f_k(x_1, \dots, x_q)) = g_k(\alpha(x_1), \dots, \alpha(x_q)).$$

The algebraic systems A_1 and A_2 are said to be **isomorphic**, and for virtually all the purposes of algebra, isomorphic systems are equivalent.

In algebraic specification, we implicitly use this same idea. We do not identify or provide a representation for individual items (members of the sorts). They are deliberately left abstract to permit the greatest variety of realizations. We know their essential properties, but no more. An ADT specification is regarded as describing any system that possesses these properties — any system *isomorphic* to the initial algebra (if that's our interpretation).

As an example, consider the usual natural numbers. We work and think of this system in its decimal representation — $5 + 6 = 11$. But computers invariably operate in binary, a convenient (for them) renaming. The computer has corresponding values (e.g., 101 and 110) and a corresponding operation, let's write binary addition as \oplus . Then $101 \oplus 110 = 1011$, or writing out the correspondence, $\alpha(5) \oplus \alpha(6) = \alpha(5+6)$. Through this isomorphism, it is irrelevant which system the operations are performed in — as long as corresponding values and operations are used, we are assured that corresponding results will be obtained.