Necessity of Hidden Functions/Sorts

Prologue

Definition: the functions explicitly included in the signature of an ADT specification are **visible** functions. Auxiliary functions added to express desired properties of visible functions are called **hidden** functions — these operations are to be inaccessible by ADT clients. We will also find it useful to add auxiliary (hidden) sorts.

Note that the no junk axiom is amended in case of the presence of hidden functions to require that each item (i.e., ground term) of each visible sort is equivalent to a ground term that includes no hidden functions.

Assertion: there are algebraic systems that have no initial or final algebra specification using only visible functions, but can be specified using hidden functions.

We establish the validity of this assertion by exhibiting an example of an algebraic system a and proving that it has no initial or final algebra specification.

Signature

The signature Σ used in our example has two sorts, N and E. With these two sorts we have four operations with signatures as follows:

$z: \to N$, $s: N \to N$
$u: \to E$, $r: N \to E$.

Note that in the ground term algebra $T(\Sigma,\varnothing)$, the sort N has values {z, s(z), s(s(z)), … }, and the sort E has values {u, r(z), r(s(z)), r(r(s(z))), … }.

Unspecifiable Algebraic System **a**

The algebraic structure a to be considered has the signature Σ given above and carrier sets Nat = {0, 1, 2, … } for N, and a copy of the even numbers plus a copy of 1, $\{1'\} \cup$ Evens' = { 1', 0', 2', 4', 6', … } for E. For operations take z = 0, u = 1', s(n) = n+1, and r(n) = **if** n even **then** n' **else** 1'.

Note that the system a has no junk — each value corresponds to a ground term in the term algebra determined by the signature.

Assertion: a has no initial or final algebra specification using only the visible functions of the signature.

Note that there are no pre-defined types that can be used to distinguish elements of N or E so in any specification all elements are indistinguishable and fall in one class. Hence the final algebra for a specification is trivial with each sort consisting of a single element, and this is obviously different from a.

Now consider an initial algebra specification for a, say $<\Sigma,e>$.

Claim 1: e can have no nontrivial equations (i.e., other than t = t for a term t) of sort N.

By the signature the only options for equations of sort N are of the form $s^m(x) = s^n(y)$ where either $m \neq n$ or $x \neq y$ (or both). Either of these cases is inconsistent with a.

This means that the only possible equations of e have to be of sort E. The only terms of sort E are of the form u, $r(s^n(z))$, and $r(s^n(x))$ for some $n \geq 0$ and some variable x.

Claim 2: e can have no nontrivial equations of sort E of the form $r(s^m(x)) = r(s^n(y))$ where either $m \neq n$, or variable $x \neq y$ (or both).
There are two cases to consider, but again, each is inconsistent with a.

Case A: Suppose the equation is $r(s^m(x)) = r(s^n(x))$ for $m \neq n$.

If m is even, take x = z and we infer that $r(s^m(z)) = r(s^n(z))$, but in a $r(s^m(x)) = r(m) = m'$, and either $r(s^n(z)) = n' \neq m'$, or $r(s^n(z)) = 1' \neq m'$. Similarly if m is odd, take x= s(z) and we infer that $r(s^{m+1}(z)) = r(s^{n+1}(z))$, but in a $r(s^{m+1}(x)) = r(m+1) = (m+1)'$, and either $r(s^{n+1}(z)) = (n+1)' \neq (m+1)'$, or $r(s^{n+1}(z)) = 1' \neq (m+1)'$.

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Case B: Suppose the equation is $r(s^m(x)) = r(s^n(y))$ for $x \neq y$.

Take $x = z$, so $r(s^m(x)) = r(s^m(z))$, and take $y = z$ or $y = s(z)$ so that $r(s^n(y)) = r(s^p(z))$ where the parity of p is the opposite of that of m, and hence the equation fails in a.

Therefore by Claim 1 and Claim 2, for an initial algebra specification $<\Sigma,e>$ for a, equations of e contain no variables and are either of the form

$r(s^m(z)) = u$, or

$r(s^m(z)) = r(s^n(z))$ for $m \neq n$.

Take M to be the first odd number strictly larger than any exponent occurring in e. Then $r(s^M(z))) = 1'$ in a, but it is impossible to infer this from e since there is no equation with a sufficiently large "exponent" to match M. Hence we conclude that there is no initial algebra specification for a.

Specification of a with Hidden Functions

Lastly, an initial algebra specification of a is possible if we permit hidden functions.

Assertion: a has an initial algebra specification when hidden functions are permitted.
Add the following hidden functions to the signature:

even: $N \rightarrow N$
testz: $N \times E \times E \rightarrow E$

with the equations (for all $w \in N$, and $x,y \in E$)

even(z) = z,
even(s(z)) = s(z)
even(s(s(w)) = even(w)

testz(z,x,y) = x
testz(s(z),x,y) = y
testz(w,x,y) = testz(even(w),x,y)
r(w) = testz(even(w),r(w),u)

Notice that instances of the function 'even' can be removed from ground terms by using its equations. Also, since 'even' applications are always equal to either z or s(z), instances of the function 'testz' can be eliminated as well. Hence this specification satisfies the no junk axiom. The term algebra equivalence classes for this extended hidden function signature are in isomorphic correspondence to a as follows:

$0 \leftrightarrow [z] = \{z, even(z), even(s^2(z)), \dots , even(s^{2n}(z)), \dots \}$
$1 \leftrightarrow [s(z)] = \{s(z), even(s(z)), even(s^3(z)), \dots , even(s^{2n+1}(z)), \dots \}$
$2 \leftrightarrow [s^2(z)] = \{s^2(z)\}$
$3 \leftrightarrow [s^3(z)] = \{s^3(z)\}$
…
$1' \leftrightarrow [u] = \{u, r(s(z)), r(s^3(z)), \dots , r(s^{2n+1}(z)) , \dots \}$
$0' \leftrightarrow [r(z)] = \{r(z)\}$
$2' \leftrightarrow [r(s^2(z))] = \{r(s^2(z))\}$
…

The difficulty in creating an initial algebra specification for a is in capturing the desired behavior for the function 'r'. Adding the hidden functions provides additional terms in the equivalence classes that permit the expression of this behavior.