MATH:6010, Spring 2019 — Prof. Bleher

Homework Set 5

THIS PROBLEM SET CHANGED ON THURSDAY, APRIL 4, 2019: PROBLEM #3 WAS ADDED. Please observe the NEW DUE DATE!

Please hand in the following problems at the **beginning of the lecture on Thursday, April 18, 2019**.

Problems:

For all problems, until stated otherwise, a *ring* is a commutative ring with 1, and a *subring* has the same 1 as the ring that contains it.

(1) (a) Give an example of an integral domain that is not integrally closed.

(b) Let R be a ring, let A be a subring, and let $x \in R$ be a unit in R. Show that every

$$y \in A[x] \cap A[x^{-1}]$$

is integral over A. (Show that there exists a nonnegative integer n such that the A-module $M = A + Ax + \cdots + Ax^n$ satisfies $yM \subseteq M$. Now argue with matrices and determinants.)

(2) Let $\alpha \in \mathbb{R}$ be such that $\alpha^3 = 2$, let $K = \mathbb{Q}(\alpha)$, and let $\mathcal{O}_K$ be the ring of integers of K.

(a) Let $z = a + b\alpha + c\alpha^2 \in K$ $(a, b, c \in \mathbb{Q})$. Determine the trace, the norm and the irreducible polynomial of z over $\mathbb{Q}$ in terms of a, b, c.

(b) Let $z = a + b\alpha + c\alpha^2 \in \mathcal{O}_K$ $(a, b, c \in \mathbb{Q})$. By calculating the traces of $z, \alpha z, \alpha^2 z$ show that $6\,\mathcal{O}_K \subseteq \mathbb{Z}[\alpha]$.

(c) Use that $z \in K$ lies in $\mathcal{O}_K$ if and only if $\mathrm{Irr}(z, \mathbb{Q}, x) \in \mathbb{Z}[x]$, together with the computations in (a) and (b) to show that $\mathcal{O}_K = \mathbb{Z}[\alpha]$.

(d) Compute the discriminant of K.

(3) Let K be a field.

(a) Show that $K[x, y]$ (the polynomial ring over K in two commuting variables) is not a Dedekind domain.

(b) Let A be the subring $K[x^2, x^3]$ of the polynomial ring $K[x]$. By using that $K[x^2] \subset A \subset K[x]$, show that A is a Noetherian ring and every non-zero prime ideal is maximal. By considering the element x, show that A is not a Dedekind domain.

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(originally posted on Mar 28 2019)
Problem #3 ADDED on Apr 4 2019