MATH:6010, Spring 2019 — Prof. Bleher

Homework Set 4

Please hand in the following problems at the **beginning of the lecture on Thursday, March 7, 2019**.

Problems:

(1) Let $\alpha \in \mathbb{R}^+$ satisfy $\alpha^2 = (9 - 5\sqrt{3})(2 - \sqrt{2})$. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \alpha)$.

(a) Prove that $K/\mathbb{Q}$ is Galois and determine $f(x) = \mathrm{Irr}(\alpha, \mathbb{Q}, x)$.

Hint: To prove normality, think about where each embedding $\tau : K \to \mathbb{Q}^a$ over $\mathbb{Q}$ can possibly send α. This consideration should also help you determine $f(x) = \mathrm{Irr}(\alpha, \mathbb{Q}, x)$.

(b) Prove that $K = \mathbb{Q}(\alpha)$. Find a set of generators of $G = \mathrm{Gal}(K/\mathbb{Q})$ and write down each of these generators as a permutation of the roots of $f(x) = \mathrm{Irr}(\alpha, \mathbb{Q}, x)$. Determine the isomorphism type of G.

(2) Let K be a cyclic extension of a field F of characteristic $p > 0$. Let $G = \mathrm{Gal}(K/F) = \langle\sigma\rangle$ be of order p^{m-1} where $m \geq 2$. Let $\beta \in K$ with $\mathrm{Tr}_{K/F}(\beta) = 1$.

(a) Show that there exists an element $\alpha \in K$ with $\sigma(\alpha) - \alpha = \beta^p - \beta$.

Hint: Use a similar element α as in the proof of Hilbert's Theorem 90, additive form, for some appropriate θ.

(b) Show that $f(x) = x^p - x - \alpha$ is irreducible in $K[x]$, where α is as in part (a).

Hint: By Artin-Schreier it is enough to show that $f(x)$ has no roots in K.

(c) Let δ be a root of $f(x)$. Show that $K(\delta)/F$ is cyclic Galois of degree p^m, and $\mathrm{Gal}(K(\delta)/F)$ is generated by an extension σ^* of σ such that $\sigma^*(\delta) = \delta + \beta + j$ for some j in the prime subfield $\mathbb{F}_p$ of F.

Hint: One possibility to prove this is by using Artin's Theorem (Theorem 1.8 of Chapter VI.1 in Lang).

Do also the following problem, but do not hand it in:

(3) In each of the following cases (a) and (b), you are given a polynomial $f(x) \in F[x]$ as follows:

(a) $f(x) = x^4 - 4x^2 - 1$, $F = \mathbb{Q}$.

(b) $f(x) = (x^3 - 2)(x^3 - 3)(x^2 - 2)$, $F = \mathbb{Q}(\sqrt{-3})$.

Let K be the splitting field of $f(x)$ over F and let $G = \mathrm{Gal}(K/F)$.

For both cases (a) and (b): Determine K and find $[K : F]$. Find a set of generators of G and write down each of these generators as a permutation of the roots of $f(x)$. Determine the isomorphism type of G.

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