MATH:6010, Spring 2019 — Prof. Bleher

Homework Set 3

Please hand in the following problems at the **beginning of the lecture on Thursday, February 21, 2019**.

Problems:

(1) Let p be a prime, and let $K = \mathbb{F}_p(u,t)$, where u and t are **algebraically independent over** $\mathbb{F}_p$. This means that there is no non-zero polynomial $f(X_1, X_2) \in \mathbb{F}_p[X_1, X_2]$ with $f(u,t) = 0$. In other words, $\mathbb{F}_p[u,t]$ is isomorphic to the polynomial ring $\mathbb{F}_p[X_1, X_2]$.

Define $E = \mathbb{F}_p(u^p - u, t^p - u)$, and let K_E be the maximal separable subextension of K/E, i.e. $K_E = \{\alpha \in K \mid \alpha \text{ separable over } E\}$.

(a) Determine the following degrees and separable degrees:

- $[K : \mathbb{F}_p(u,t^p)]$ and $[K : \mathbb{F}_p(u,t^p)]_s$
- $[\mathbb{F}_p(u,t^p) : E]$ and $[\mathbb{F}_p(u,t^p) : E]_s$
- $[K : E]$ and $[K : E]_s$

Describe K_E as a finitely generated field extension of $\mathbb{F}_p$, i.e find finitely many $\beta_1, \ldots, \beta_n \in K_E$ such that $K_E = \mathbb{F}_p(\beta_1, \ldots, \beta_n)$.

Hint: To show that certain polynomials are irreducible over the respective fields, use the algebraic independence of u and t over $\mathbb{F}_p$.

(b) Find an element $\alpha \in K$ such that $[E(\alpha) : E] = p$ and $[E(\alpha) : E]_s = 1$. In other words, $E(\alpha)$ is a simple extension of E that is not separable over E.

Hint: Let $v = u^p - u$, and $w = t^p - u$, and consider $v - w$. This is not the α but related to it.

(2) This exercise completes what we have started in #1 of Homework 2.

Let E/F be a field extension and assume that every non-constant polynomial in $F[x]$ has a root in E. Prove that every non-constant polynomial in $F[x]$ splits into linear factors in $E[x]$.

Hint: If F is a perfect field, use #1 of Homework 2. If F is not perfect, let $F_0 = \{\alpha \in E \mid \alpha \text{ is purely inseparable over } F\}$. Show that F_0 is a subfield of E containing F and that F_0 is perfect. Use F_0 to complete the proof.

Conclude: If E/F is an **algebraic** field extension, then E is algebraically closed if and only if every non-constant polynomial in $F[x]$ has a root in E.

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