MATH:6010, Spring 2019 — Prof. Bleher

Homework Set 2

Please hand in the following problems at the **beginning of the lecture on Thursday, February 7, 2019**.

Problems:

(1) Let F be a field such that F has the following property:
($*$) Every finite extension of F is simple.
(We will prove later that if F is perfect (e.g. if $\mathrm{char}(F) = 0$ or if F is a finite field) then F has property ($*$).)

(a) Let E/F be a field extension and assume that every non-constant polynomial in $F[x]$ has a root in E. Prove that every non-constant polynomial in $F[x]$ splits into linear factors in $E[x]$.

Hint: Let $f(x) \in F[x]$ be non-constant, let F^a be an algebraic closure of F, let $\alpha_1, \ldots, \alpha_k$ be the distinct roots of $f(x)$ in F^a and let $K = F(\alpha_1, \ldots, \alpha_k)$ (i.e. K is a splitting field of $f(x)$ over F). By ($*$), K is a simple extension of F. Let $\gamma \in K$ be a primitive element and consider $g(x) = \mathrm{Irr}(\gamma, F, x)$. Now use that $g(x)$ has a root in E.

(b) Conclude: If F is a field satisfying ($*$) and E/F is an **algebraic** extension, then E is algebraically closed if and only if every non-constant polynomial in $F[x]$ has a root in E. When we drop the assumption that E/F is algebraic, show that this may not be true (give an example).

(2) Let F be a field of characteristic 0, let F^a be a fixed algebraic closure of F, and let $f(x) \in F[x]$ be a monic irreducible quadratic polynomial. For $n \in \mathbb{Z}^+$, define

$$f_1(x) = f(x) \quad \text{and} \quad f_n(x) = f(f_{n-1}(x)) = f_{n-1}(f(x)) \quad \text{for } n \geq 2.$$

(a) Let $n \in \mathbb{Z}^+$. Determine the degree of $f_n(x)$. Let $K_n \subseteq F^a$ be a splitting field of $f_n(x)$ over F. Show that $K_n \subseteq K_{n+1}$.

(b) Prove that every embedding $\sigma : K_{n+1} \to F^a$ over K_n induces an automorphism of K_{n+1} of order 1 or 2.

(3) Let F be a field, let $f(x)$ be in $F[x]$ of degree $n \geq 1$, and let K be a splitting field of $f(x)$ over F. Show that $[K : F]$ divides $n!$

Hint: Use induction, and distinguish the cases of $f(x)$ being irreducible and reducible in $F[x]$.

Do also the following problems, but do not hand them in:

(4) Suppose F is a field of positive characteristic p. Let $a \in F$ and suppose a has no p^{th} root in F. Prove that for all $n \in \mathbb{Z}^+$, $t^{p^n} - a$ is an irreducible polynomial in $F[t]$. Please prove this using elementary arguments (use characteristic p and consider the constant coefficients of monic divisors of $t^{p^n} - a$).

(5) Here are a couple of problems to get you back into computations with fields. You should justify all your answers.
   (a) Find a splitting field K of $x^4 + 4$ over $\mathbb{Q}$ and determine $[K : \mathbb{Q}]$.
   (b) Find a splitting field K of $x^{12} - 9$ over $\mathbb{Q}$ and determine $[K : \mathbb{Q}]$.
   (c) Find a splitting field K of $(x^4 + x^2 + 1)(x^2 + 3)$ over $\mathbb{Q}$ and determine $[K : \mathbb{Q}]$.
   (d) Let $\gamma = \sqrt[4]{11} \in \mathbb{R}^+$. Determine $[\mathbb{Q}(\gamma + i\gamma) : \mathbb{Q}]$. Is $\mathbb{Q}(\gamma + i\gamma)/\mathbb{Q}$ normal?

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