MATH:6010, Spring 2019 — Prof. Bleher

Homework Set 1

Please hand in the following problems at the **beginning of the lecture on Thursday, January 24, 2019**.

Problems:

(1) A field F is said to be *formally real* if -1 is not expressible as a sum of squares in F. Let F be a formally real field, let $f(x) \in F[x]$ be an irreducible polynomial of odd degree and let α be a root of $f(x)$. Prove that $F(\alpha)$ is formally real.

Hint: Let α be a counterexample of minimal degree. Show that $-1 + f(x)g(x) = (p_1(x))^2 + \cdots + (p_m(x))^2$ for some $p_i(x), g(x) \in F[x]$ where $g(x)$ has odd degree $<$ $\deg f$. Show that some root β of g has odd degree over F and $F(\beta)$ is not formally real, contradicting the minimality of α.

(2) Let F be a field and let $E = F(x)$ where x is transcendental over F, i.e. x is not algebraic over F. In other words, E is the field of rational functions in x with coefficients in F.

(a) Let $K \subseteq E$ be a subfield of E such that K properly contains F. Prove that x is algebraic over K.

(b) Let $y = f(x)/g(x)$ be in $E = F(x)$, where $f(x)$ and $g(x)$ are relatively prime polynomials in $F[x]$. Let $n = \max(\deg f, \deg g)$, and suppose $n \geq 1$. Prove $[F(x) : F(y)] = n$.

Hint: Gauss' Lemma may be useful.

Do also this problem, but do not hand it in:

(3) Let F be a field of characteristic $\neq 2$. Let D_1, D_2 be elements of F, neither of which is a square in F.

(a) Prove that the field $F(\sqrt{D_1}, \sqrt{D_2})$ is of degree 4 over F if $D_1 \cdot D_2$ is not a square in F and that it is of degree 2 over F otherwise. When the field $F(\sqrt{D_1}, \sqrt{D_2})$ is of degree 4 over F, it is called a biquadratic extension of F.

(b) Assume that $D_1 \cdot D_2$ is not a square in F.
Prove that $F(\sqrt{D_1} + \sqrt{D_2}) = F(\sqrt{D_1}, \sqrt{D_2})$ and find the monic irreducible polynomial over F satisfied by $\sqrt{D_1} + \sqrt{D_2}$.

Hint: Look at powers of $\sqrt{D_1} + \sqrt{D_2}$.

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