MATH:6010, Spring 2019 — Prof. Bleher

BONUS HOMEWORK SET

This homework is a bonus homework in the sense that you can use it to replace one other homework. **COMPLETE 2 OF THE 4 PROBLEMS.** If you complete more than 2 problems, please say which 2 problems should count toward this homework.

CHANGE OF DUE DATE: As of Thursday, April 4, 2019, IF you want this homework to count toward your grade, this bonus homework is now due on THURSDAY, April 11, 2019, at the beginning of the lecture.

Let K/F be an infinite Galois extension with $G = \mathrm{Gal}(K/F)$. Let

$$\mathcal{E} = \{E \mid E/F \text{ is a finite Galois extension with } E \subset K\}.$$

Then $\mathcal{E}$ is partially ordered by inclusion, and directed since if E_1 and E_2 are in $\mathcal{E}$, then E_1E_2 is in $\mathcal{E}$. We obtain that $\left(\{\mathrm{Gal}(E/F)\}_{E\in\mathcal{E}}, \{f_E^{E'}\}_{E\subseteq E'}\right)$ is an inverse system where $f_E^{E'}$ is given by restriction, i.e. $f_E^{E'} : \mathrm{Gal}(E'/F) \to \mathrm{Gal}(E/F)$ is defined by $f_E^{E'}(\sigma) = \sigma\big|_E$ if $E \subseteq E'$ in $\mathcal{E}$. We have proved in class that the group homomorphism

$$\begin{array}{rccl} \rho: & G & \to & \varprojlim_{E\in\mathcal{E}} \mathrm{Gal}(E/F) \\ & \sigma & \mapsto & \left(\sigma\big|_E\right)_{E\in\mathcal{E}} \end{array}$$

is an isomorphism.

I will use the following **conventions** for this homework: **Neighborhoods are assumed to be open. A neighborhood basis of a point x consists of a collection $\mathcal{B}$ of neighborhoods of x such that for every neighborhood U of x there exists a $U' \in \mathcal{B}$ with $U' \subseteq U$. A topological space is said to be compact if every open covering has a finite subcovering.**

Define the **Krull topology** on $G = \mathrm{Gal}(K/F)$ as follows: For each $\sigma \in G$, a neighborhood basis is given by $\{\sigma\,\mathrm{Gal}(K/N) \mid N/F \text{ is a finite extension with } N \subset K\}$. (Note that N/F need not be Galois.)

We have proved in class: For each $\sigma \in G$, $\{\sigma\,\mathrm{Gal}(K/E)\}_{E\in\mathcal{E}}$ is also a neighborhood basis of σ in the Krull topology.

COMPLETE 2 OF THE FOLLOWING 4 PROBLEMS:
(#3 and #4 are the more interesting problems)

(1) Let $\sigma \in G$ and let $E \in \mathcal{E}$. Define $U_E(\sigma) = \{\tau \in G \mid \tau\big|_E = \sigma\big|_E\}$. Explain the following (easy) facts (a brief explanation for each of them is enough):

  (a) We have $U_E(\sigma) = \sigma U_E(\mathrm{id}) = \sigma\mathrm{Gal}(K/E)$; hence $\{U_E(\sigma)\}_{E\in\mathcal{E}}$ is a neighborhood basis of σ in G.

  (b) If $E = F(x_1, \ldots, x_n)$, then $U_E(\sigma) = \{\tau \in G \mid \tau(x_i) = \sigma(x_i) \text{ for all } 1 \le i \le n\}$.

(c) A subgroup $A \leq G$ is open if and only if for all $\sigma \in A$ there exists $E \in \mathcal{E}$ such that $U_E(\sigma) \subseteq A$. The closure of A is

$$\overline{A} = \{\sigma \in G \mid \text{ for all } E \in \mathcal{E} \text{ there exists } \tau \in A \text{ with } \tau\big|_E = \sigma\big|_E\}.$$

(d) Let A be a subgroup of G and let $\overline{A}$ be its closure. Then the corresponding fixed fields satisfy $K^A = K^{\overline{A}}$.

(e) We have $U_E(\mathrm{id}) = \mathrm{Ker}\left(G \xrightarrow{\mathrm{res}_E} \mathrm{Gal}(E/F)\right)$ where $\mathrm{res}_E(\sigma) = \sigma\big|_E$ for all $\sigma \in G$. Moreover, res_E is continuous when $\mathrm{Gal}(E/F)$ is given the discrete topology. Conclude that $\mathrm{Gal}(K/E)$ is open and closed and normal of finite index in G.

(2) The inverse limit $\varprojlim \mathrm{Gal}(E/F)$ is embedded into the direct product

$$\varprojlim_{E \in \mathcal{E}} \mathrm{Gal}(E/F) \to \prod_{E \in \mathcal{E}} \mathrm{Gal}(E/F).$$

Give $\mathrm{Gal}(E/F)$ the discrete topology for all $E \in \mathcal{E}$, and give the direct product the product topology. By Tychonoff's Theorem, the direct product is compact.

Prove that the inverse limit $\varprojlim \mathrm{Gal}(E/F)$ is closed in the product, and therefore compact. Prove that the isomorphism $\rho : G \to \varprojlim \mathrm{Gal}(E/F)$ is continuous and open. Conclude that G is also compact.

(3) Let $F \subset N \subset K$ be a subextension. Prove that $\mathrm{Gal}(K/N)$ is closed in $G = \mathrm{Gal}(K/F)$, and the Krull topology on $\mathrm{Gal}(K/N)$ is induced by the Krull topology on G.

Hint: For all $x \in N$, consider the map $f_x : G \to K$ defined by $f_x(\sigma) = \sigma(x)$. Giving K the discrete topology, show that f_x is continuous. Describe the preimage of $\{x\}$ under f_x for all x, and write $\mathrm{Gal}(K/N)$ in terms of these preimages.

(4) Let $\Delta \leq G$ be a subgroup, and let $N = K^\Delta$ be the corresponding fixed field. Prove that $\mathrm{Gal}(K/N) = \overline{\Delta}$.

Hint: It is not hard to see that $\overline{\Delta} \subseteq \mathrm{Gal}(K/N)$. For the other inclusion, let $\sigma \in \mathrm{Gal}(K/N)$ and let $x_1, \ldots, x_n \in K$ be arbitrary. Let K_0 be the Galois closure of $N(x_1, \ldots, x_n)$ over N. Then $K_0 \subseteq K$ (why?). Consider the restriction homomorphism $\mathrm{res}_{K_0} : \mathrm{Gal}(K/N) \to \mathrm{Gal}(K_0/N)$ and let Δ_0 be the image of Δ under this homomorphism. Prove that $N = K_0^{\Delta_0}$ and conclude that $\mathrm{Gal}(K_0/N) = \Delta_0$, and hence $\sigma\big|_{K_0} \in \Delta_0$. Now use #1(c) to conclude that $\sigma \in \overline{\Delta}$.

Note: Exercises #3 and #4 prove that the fundamental theorem of Galois theory is also true for infinite Galois extensions as long as we restrict ourselves to the closed subgroups of $\mathrm{Gal}(K/F)$ with respect to the Krull topology. In other words, the subextensions of K/F are in bijective correspondence with the closed subgroups of $\mathrm{Gal}(K/F)$.

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