

HOMEWORK SET 6

Please hand in the following problems at the **beginning of the lecture on Thursday, November 1**. Problems (3) and (4) are good to know but you do not have to hand them in.

**Problems:** All rings are assumed to have a 1 and all modules are assumed to be unital.

(1) Prove **either (a) or (b)**:

(a) The ring  $S = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \mid a \in \mathbb{Z}, b, c \in \mathbb{Q} \right\}$  is left Noetherian but not right Noetherian.

(b) The ring  $T = \left\{ \begin{bmatrix} r & s \\ 0 & t \end{bmatrix} \mid r \in \mathbb{Q}, s, t \in \mathbb{R} \right\}$  is right Artinian but not left Artinian.

(2) Let  $R$  be a commutative ring. Prove that  $R$  is Noetherian if and only if every prime ideal is finitely generated.

**Hint:** One direction is obvious. For the other direction, you may use the following steps: Prove first that IF the collection of ideals of  $R$  that are NOT finitely generated is not empty, then it contains a maximal element  $I$ , and that  $R/I$  is a Noetherian ring. Using that  $I$  cannot be a prime ideal (why?), prove that there exist finitely generated ideals  $J_1, J_2$  containing  $I$  with  $J_1 J_2 \subseteq I$  and that  $J_1 J_2$  is finitely generated. Then prove that  $I/J_1 J_2$  is a finitely generated  $R/I$ -submodule of  $J_1/J_1 J_2$ . Show that this gives a contradiction.

(3) Let  $R$  be a commutative ring and let  $\mathfrak{m}$  be a maximal ideal of  $R$ . Suppose there exists a positive integer  $n$  with  $\mathfrak{m}^n = 0$ . Prove that  $R$  is Noetherian if and only if  $R$  is Artinian.

**Hint:** Consider the descending chain of ideals  $R \supseteq \mathfrak{m} \supseteq \cdots \supseteq \mathfrak{m}^{n-1} \supseteq \mathfrak{m}^n = 0$ , and note that each successive quotient  $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ ,  $0 \leq i \leq n-1$ , is a vector space over the field  $F = R/\mathfrak{m}$ .

(4) Let  $R$  be a ring with Jacobson radical  $J$ , and let  $M$  be a finitely generated left  $R$ -module. Prove that if  $x_1, \dots, x_n$  are elements of  $M$  whose images in  $M/JM$  generate  $M/JM$ , then they generate  $M$ .