

HOMEWORK SET 5

Please hand in the following problems at the **beginning of the lecture on Tuesday, October 23.**

Problems: All rings are assumed to have a 1 and all modules are assumed to be unital.

- (1) (a) Let D be a nonzero divisible $\mathbb{Z}$ -module. Prove that D is not a projective $\mathbb{Z}$ -module. In particular, this shows that $\mathbb{Q}$ is a torsionfree $\mathbb{Z}$ -module that is not projective.

Hint: Show first that if F is a free $\mathbb{Z}$ -module then $\bigcap_{n=1}^{\infty} nF = 0$ (use a basis of F to prove this). Then use the criterion for projective modules involving free modules.

- (b) Let k be a field, and let $R = k[x, y]$ be the polynomial ring over k in two commuting variables x and y . Let $Q = k(x, y)$ be the fraction field of R , and let $M = Q/R$. Prove that M is a divisible R -module that is not an injective R -module.

Hint: To show that M is not injective, let $I = Rx + Ry$, i.e. I is the (left) ideal of R generated by x and y . Consider the map $\alpha : I \rightarrow M = Q/R$ defined by

$$\alpha(f(x, y)) = \frac{f(x, 0)}{xy} + R$$

for all $f(x, y) \in I$. Show that α is an R -module homomorphism which cannot be extended to an R -module homomorphism $\beta : R \rightarrow M$.

- (2) Let M be a right R -module. Prove that the following statements are equivalent:

- (i) M is flat and $M \otimes_R X$ is non-zero for every non-zero left R -module X .
- (ii) A sequence $N' \xrightarrow{\alpha} N \xrightarrow{\beta} N''$ in $R\text{-Mod}$ is exact if and only if the resulting sequence $M \otimes_R N' \xrightarrow{\text{id}_M \otimes \alpha} M \otimes_R N \xrightarrow{\text{id}_M \otimes \beta} M \otimes_R N''$ in $\mathbb{Z}\text{-Mod}$ is exact.

Hint: Note that M is flat if and only if the exactness of a sequence $N' \xrightarrow{\alpha} N \xrightarrow{\beta} N''$ in $R\text{-Mod}$ always implies the exactness of the resulting sequence $M \otimes_R N' \xrightarrow{\text{id}_M \otimes \alpha} M \otimes_R N \xrightarrow{\text{id}_M \otimes \beta} M \otimes_R N''$ in $\mathbb{Z}\text{-Mod}$. Why is this true?