

HOMEWORK SET 4

Please hand in problems (1) and (2) at the **beginning of the lecture on Tuesday, October 2**. Problems (3) and (4) are good to know but you do not have to hand them in.

Problems: All rings are assumed to have a 1 and all modules are assumed to be unital.

- (1) Let R be a commutative ring with 1, let G be a finite group, and let RG be the group ring of G over R . Recall that each element $x \in RG$ can be written uniquely in the form $x = \sum_{g \in G} \lambda_g g$ with $\lambda_g \in R$ for all $g \in G$.

A *class sum* in RG is defined to be the sum of the elements in a conjugacy class of G , i.e. if $\mathcal{K} = \{k_1, \dots, k_m\}$ is a conjugacy class in G , the corresponding class sum is $K = k_1 + \dots + k_m$.

- (a) Prove that the class sums in RG lie in the center of RG .
- (b) Let $\mathcal{K}_1, \dots, \mathcal{K}_r$ be the distinct conjugacy classes of G , and let $K_1, \dots, K_r$ be the corresponding class sums. Prove that an element $\alpha \in RG$ lies in the center of RG if and only if $\alpha = a_1 K_1 + a_2 K_2 + \dots + a_r K_r$ for some $a_1, \dots, a_r \in R$.
- (c) Prove that the center of RG is a free R -module on the set $\{K_1, \dots, K_r\}$.
- (2) Let R be a commutative ring with 1, let I be an ideal of R , and let M and N be R -modules. Show that M/IM and N/IN have a natural structure as R -modules and as (R/I) -modules. Prove that

$$(M \otimes_R N)/I(M \otimes_R N) \cong (M/IM) \otimes_{R/I} (N/IN) \quad \text{as } R\text{-modules.}$$

Recall that if X is an R -module, then

$$IX = \{r_1 x_1 + \dots + r_n x_n \mid n \in \mathbb{Z}^+, r_i \in I, x_i \in X \text{ for all } 1 \leq i \leq n\}.$$

- (3) This exercise shows that free modules over non-commutative rings need not have a unique rank.

For each $i \in \mathbb{Z}^+$, let M_i be the free $\mathbb{Z}$ -module $\mathbb{Z}$ (of rank 1). Define M to be the direct product $M = \prod_{i \in \mathbb{Z}^+} M_i$, so each element of M can be written uniquely in the form $(a_1, a_2, a_3, \dots)$ with $a_i \in \mathbb{Z}$ for all $i \in \mathbb{Z}^+$. Let R be the endomorphism ring of M , i.e. $R = \text{End}_{\mathbb{Z}}(M)$. Define $\varphi_1, \varphi_2 \in R$ by

$$\begin{aligned} \varphi_1((a_1, a_2, a_3, \dots)) &= (a_1, a_3, a_5, \dots) \\ \varphi_2((a_1, a_2, a_3, \dots)) &= (a_2, a_4, a_6, \dots) \end{aligned}$$

Prove that R is a free left R -module on the set $\{\varphi_1, \varphi_2\}$. Use this to prove that $R \cong R^2$ as left R -modules. Deduce that $R \cong R^n$ for all $n \in \mathbb{Z}^+$.

Hint: Define two maps $\psi_1, \psi_2 \in R$ by $\psi_1((a_1, a_2, \dots)) = (a_1, 0, a_2, 0, \dots)$ and $\psi_2((a_1, a_2, \dots)) = (0, a_1, 0, a_2, \dots)$. Show that $\varphi_i \psi_i = 1$, $\varphi_1 \psi_2 = 0 = \varphi_2 \psi_1$ and $\psi_1 \varphi_1 + \psi_2 \varphi_2 = 1$ (where 1 denotes the identity endomorphism in R). Use this to prove that φ_1, φ_2 are linearly independent over R and generate R as a left R -module.

(4) Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$. The ring $R/I \cong \mathbb{Z}/2\mathbb{Z}$ is naturally an R -module on which both 2 and x act as zero.

(a) Show that there is an R -module homomorphism $\varphi : I \otimes_R I \rightarrow \mathbb{Z}/2\mathbb{Z}$ with

$$\varphi(p(x) \otimes q(x)) = \frac{p(0)}{2} q'(0) \pmod{2}$$

for all $p(x), q(x) \in I$, where $q'(x)$ denotes the polynomial derivative of $q(x)$. Use this to show that $2 \otimes x \neq x \otimes 2$ in $I \otimes_R I$.

(b) Show that there is a surjective R -module homomorphism $\psi : I \otimes_R I \rightarrow I^2$ with $\psi(p(x) \otimes q(x)) = p(x)q(x)$ for all $p(x), q(x) \in I$. Use this to show that the element $2 \otimes 2 + x \otimes x$ in $I \otimes_R I$ is not a simple tensor, i.e. it cannot be written as $a(x) \otimes b(x)$ for some $a(x), b(x) \in I$.