

HOMEWORK SET 3

Please hand in the following problems at the **beginning of the lecture on Thursday, September 20.**

Problems:

In #1, we use the following notation: Let $\mathcal{C}$ be a category, and let $C_1, C_2, X \in \text{Ob}(\mathcal{C})$. Suppose the product $(C_1 \amalg C_2, \{p_1, p_2\})$ exists. If there are morphisms $g_i : X \rightarrow C_i$ for $i \in \{1, 2\}$, denote the unique morphism $h \in \text{Hom}_{\mathcal{C}}(X, C_1 \amalg C_2)$ with $p_i \circ h = g_i$ by (g_1, g_2) .

- (1) Let $\mathcal{C}$ be a concrete category having finite products and a couniversal object Ω . A *group object* in $\mathcal{C}$ is a quadruple (G, μ, η, ϵ) , where G is an object in $\mathcal{C}$, and $\mu : G \amalg G \rightarrow G$, $\eta : G \rightarrow G$, and $\epsilon : \Omega \rightarrow G$ are morphisms in $\mathcal{C}$, so that the following diagrams commute:

Associativity:

$$\begin{array}{ccc} G \amalg G \amalg G & \xrightarrow{\text{id}_G \amalg \mu} & G \amalg G \\ \mu \amalg \text{id}_G \downarrow & & \downarrow \mu \\ G \amalg G & \xrightarrow{\mu} & G \end{array}$$

Identity:

$$\begin{array}{ccccc} G \amalg \Omega & \xrightarrow{\text{id}_G \amalg \epsilon} & G \amalg G & \xleftarrow{\epsilon \amalg \text{id}_G} & \Omega \amalg G \\ & \searrow \lambda & \downarrow \mu & \swarrow \rho & \\ & & G & & \end{array}$$

where $\lambda : G \amalg \Omega \rightarrow G$ and $\rho : \Omega \amalg G \rightarrow G$ are projection morphisms.

Inverse:

$$\begin{array}{ccccc} G & \xrightarrow{(\text{id}_G, \eta)} & G \amalg G & \xleftarrow{(\eta, \text{id}_G)} & G \\ \omega \downarrow & & \downarrow \mu & & \downarrow \omega \\ \Omega & \xrightarrow{\epsilon} & G & \xleftarrow{\epsilon} & \Omega \end{array}$$

where $\omega : G \rightarrow \Omega$ is the unique morphism to the couniversal object.

- (a) Prove that a group object in **Sets** is a group.
- (b) Prove that a group object in **Grp** is an abelian group.

Hint: Use the couniversal object in **Sets**, respectively in **Grp**.

- (2) Prove that the coproduct $(\mathbb{Z}/2\mathbb{Z}) \coprod (\mathbb{Z}/3\mathbb{Z})$ in the category **Grp** is isomorphic to the modular group $\mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ (where I is the identity 2×2 matrix).

Hint: Let $G = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$. Recall from Homework Set 1, #3, that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Use the images of Y and YX in G to prove that G is isomorphic to the group

$$H = \langle a, b \mid a^2 = 1 = b^3 \rangle,$$

i.e. H is the group generated by a and b such that the ONLY relations satisfied by a and b are that a has order 2 and b has order 3. Now complete the proof of the original problem.

- (3) Let R be a ring with 1 and center $Z(R)$. Let $R\text{-Mod}$ denote the category of unital left R -modules. Let $\mathrm{Id}_{R\text{-Mod}}$ be the identity functor on $R\text{-Mod}$.

Let S be the collection of all natural transformations $\mathrm{Id}_{R\text{-Mod}} \rightarrow \mathrm{Id}_{R\text{-Mod}}$. Prove that S is a set, and in fact, a ring isomorphic to $Z(R)$, following these steps:

Define $\varphi : S \rightarrow R$ by $\varphi(\tau) = \tau_R(1)$ (where 1 is the 1 in R). Show that $\varphi(\tau) = \varphi(\tau')$ implies that $\tau = \tau'$. Use this to show that S is a set and to define a ring structure on S . Finally show that the image of φ is $Z(R)$.