

HOMEWORK SET 2

Please hand in the following problems at the **beginning of the lecture on Tuesday, September 11.**

Problems:

- (1) Let $G \neq 1$ be a group. A *minimal normal subgroup* of G is a normal subgroup $M \neq 1$ of G such that the only normal subgroups of G which are contained in M are 1 and M . A *maximal subgroup* of G is a subgroup $H \neq G$ of G such that the only subgroups of G containing H are H and G .
 - (a) Prove that every minimal normal subgroup of a finite solvable group $G \neq 1$ is an elementary abelian p -group for some prime p .
Hint: If M is a minimal normal subgroup of G , consider its characteristic subgroups M' and $\langle x^q \mid x \in M \rangle$ for all primes q .
 - (b) Prove that every maximal subgroup of a finite solvable group $G \neq 1$ has prime power index.
Hint: Prove this by induction on $\#G$. Let H be a maximal subgroup of G and let M be a minimal normal subgroup of G . Apply induction to G/M and consider separately the two cases: $M \leq H$ and $M \not\leq H$.
- (2) Let G be a group. The *Frattini subgroup* of G , denoted by $\Phi(G)$, is defined to be the intersection of all the maximal proper subgroups of G (if G has no maximal proper subgroups, set $\Phi(G) = G$). An element $x \in G$ is called a *nongenerator* if for every proper subgroup H of G , $\langle x, H \rangle$ is also a proper subgroup of G .
 - (a) Prove that $\Phi(G)$ is a characteristic subgroup of G .
 - (b) If $\#G > 1$, prove that $\Phi(G)$ is the set of nongenerators of G .
Hint: Prove first that if H is a proper subgroup of G and there exists $x \in G$ with $\langle H, x \rangle = G$, then there exists a maximal proper subgroup M of G with $H \leq M$. (Use Zorn's Lemma on the set $\mathcal{S} = \{T \mid H \leq T < G, x \notin T\}$ which is partially ordered by inclusion.)
 - (c) If G is a finite group, prove that $\Phi(G)$ is nilpotent.
Hint: Use Frattini's Argument to prove that every Sylow subgroup of $\Phi(G)$ is normal.

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Due date changed on Sep 1 2018