

HOMEWORK SET 1

Please hand in the following problems at the **beginning of the lecture on Thursday, August 30.**

Problems:

- (1) For this exercise, let G be a finite group acting on a finite set T . Let $\ell \in \mathbb{Z}^+$, $\ell \leq \#T$, and let $T^{[\ell]}$ be the set of all ℓ -tuples of elements of T with pairwise distinct entries. Then G acts in a natural way on $T^{[\ell]}$. We say the action of G on T is ℓ -transitive if given two ℓ -tuples $(t_1, \dots, t_\ell), (t'_1, \dots, t'_\ell) \in T^{[\ell]}$, there exists $\sigma \in G$ such that $\sigma.t_i = t'_i$ for all $1 \leq i \leq \ell$.

- (a) Let $n \geq 3$. Prove that the action of the alternating group A_n on $\{1, 2, \dots, n\}$ is $(n-2)$ -transitive.
- (b) Let $m \geq 2$ and assume $m \leq \#T$. Prove that G acts m -transitively on T if and only if G acts transitively on T and for all $x \in T$, the stabilizer G_x acts $(m-1)$ -transitively on $T - \{x\}$.
- (c) Suppose $n \geq 3$, and G acts faithfully on $\{1, 2, \dots, n\}$. (So G can be identified with a subgroup of S_n .) Prove that if G acts $(n-2)$ -transitively on $\{1, 2, \dots, n\}$, then $G = S_n$ or $G = A_n$.

Hint: You can use e.g. induction and part (b). Or you can try and do this more directly.

- (2) Let Ω be an infinite set and let S_Ω be the group of all permutations of Ω , i.e. $S_\Omega = \text{Sym}(\Omega)$. For $\sigma \in S_\Omega$, let $\text{Mov}(\sigma) = \{\omega \in \Omega \mid \sigma(\omega) \neq \omega\} \subseteq \Omega$. Let

$$D = \{\sigma \in S_\Omega \mid \text{Mov}(\sigma) \text{ is finite}\},$$

$$A = \{\sigma \in D \mid \sigma \text{ is an even permutation of } \text{Mov}(\sigma)\}.$$

Prove that D is a normal subgroup of S_Ω and prove that A is an infinite simple group.

- (3) Show that $\text{SL}_2(\mathbb{Z}) = \{A \in \text{Mat}_2(\mathbb{Z}) \mid \det(A) = 1\}$ is generated by the matrices

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hint: A nice way to do this is to first show the following: Suppose we have a column vector $\begin{pmatrix} a \\ c \end{pmatrix}$ where $a, c \in \mathbb{Z} - \{0\}$ are relatively prime. Then multiplying this column vector from the left with a succession of matrices of the form X, X^{-1} and Y above, we can obtain a column vector of the form $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. As you may soon see, this goes back to the Euclidean algorithm.