MATH:6000, Fall 2018 — Prof. Bleher

Take-home midterm exam

Please hand in the following problems at the beginning of the lecture on Thursday, 10/18/18.

RULES:

You should ONLY use your notes, the homework assignments, Chapters 1–6, Chapter 10 and Appendices I, II from the Dummit and Foote text and the pdf files from other books that are provided on the following website (at the dates August 21/23 and September 04/06):
`http://homepage.divms.uiowa.edu/~fbleher/m6000homework.html`
You can quote theorems and other results from the above chapters/pdf files if they are completely proved in the text or by you in a homework problem. You can also quote results from exercises which were on the Homework assignments 1-4 (including #3 and #4 on HW4).

PLEASE SOLVE THE MIDTERM PROBLEMS ON YOUR OWN and do not discuss it with anyone except me. If you have any questions, please contact me (by e-mail or during office hours).

PLEASE SIGN THE FOLLOWING STATEMENT AND TURN IT IN WITH YOUR SOLUTIONS:

I have read and understood the above rules and I have abided by them when solving the midterm problems.

Name and signature: ____

Midterm Problems: (maximal total score 100 points)

(1) Let G be a finite group.

   (a) Let A and B be nilpotent normal subgroups of G. Prove that AB is a nilpotent normal subgroup of G.
   Deduce that the product of all nilpotent normal subgroups of G is the unique maximal nilpotent normal subgroup of G. Denote this subgroup by $F(G)$.

   Hint: To prove nilpotency of AB, prove that every Sylow subgroup is normal.

   (b) Let $\Phi(G)$ be the Frattini subgroup of G (see Homework 2). Suppose N is a normal subgroup of G containing $\Phi(G)$. Prove that if $N/\Phi(G)$ is nilpotent, then N is nilpotent.

   Hint: Frattini's Argument may be useful at some point.

   (c) Prove that $\Phi(G) \leq F(G)$. If G is non-trivial and solvable, prove that $\Phi(G) < F(G)$.

   Hint: For the second statement, consider a minimal normal subgroup of $G/\Phi(G)$ and use part (b).

   (d) Prove that $F(G/\Phi(G)) = F(G)/\Phi(G)$.

(2) Let R be a commutative ring with 1, let G be a finite group, and let RG be the group ring of G over R. Recall that we have an injective ring homomorphism

$$\begin{array}{rrcl} \alpha : & R & \to & Z(RG) \\ & r & \mapsto & r\,1_G \end{array}$$

which makes RG into an R-algebra and lets us identify R with a subring of RG that is contained in the center of RG.

Let M and N be left RG-modules. Then both the tensor product $M \otimes_R N$ and the Hom group $\mathrm{Hom}_R(M, N)$ are R-modules.

(a) For all $x = \sum_{g \in G} r_g g \in RG$ and each simple tensor $m \otimes n \in M \otimes_R N$ define

$$x\,(m \otimes n) = \sum_{g \in G} r_g\,(gm \otimes gn).$$

Prove that this gives a well-defined action of RG on $M \otimes_R N$ which makes $M \otimes_R N$ into an RG-module.

(b) For all $x = \sum_{g \in G} r_g g \in RG$ and each $\varphi \in \mathrm{Hom}_R(M, N)$ and $m \in M$ define

$$(x\,\varphi)(m) = \sum_{g \in G} r_g\, g(\varphi(g^{-1}m)).$$

Prove that this gives a well-defined action of RG on $\mathrm{Hom}_R(M, N)$ which makes $\mathrm{Hom}_R(M, N)$ into an RG-module.

(c) Let $T = R$ and define for all $x = \sum_{g \in G} r_g g \in RG$ and each $t \in T$

$$x\,t = (\sum_{g \in G} r_g)\,t.$$

Prove that this gives a well-defined action of RG on T which makes T into an RG-module.

(d) **Let U and V be RG-modules, and assume that when we view U and V as R-modules via α then U and V are free R-modules of finite rank.**

Let T be as in part (c) and view $\mathrm{Hom}_R(U, T)$ as an RG-module using part (b). Prove that $\mathrm{Hom}_R(U, T) \otimes_R V$ and $\mathrm{Hom}_R(U, V)$ are isomorphic as RG-modules.

(e) **BONUS** (you do not have to hand this in): Prove part (d) under the weaker assumption that U and V are finitely generated projective when viewed as R-modules. Find a counter-example to part (d) when you do not assume projectivity.

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