MATH:5010, Spring 2017 — Prof. Bleher

Some extra problems (not to be handed in)

(1) Let F be a perfect field. Let E/F be an **algebraic** extension and assume that every non-constant polynomial in $F[x]$ has a root in E. Prove that E is algebraically closed.

Hint: Let $\overline{E}$ be an algebraic closure of E. If E is not algebraically closed, then there exists an element $\alpha \in \overline{E} - E$. Show that α is algebraic over F. Let $f(x)$ be the minimal polynomial of α over F, i.e. $f(x) = m_{\alpha,F}(x)$. Let $K \subseteq \overline{E}$ be a splitting field of $f(x)$ over F. Since F is perfect, $K = F(\gamma)$ for some $\gamma \in K$. Let $g(x) = m_{\gamma,F}(x)$. Now use the assumption that $g(x)$ has a root in E.

(2) Let F be a field of characteristic 0, let $\overline{F}$ be a fixed algebraic closure of F, and let $f(x) \in F[x]$ be a monic irreducible quadratic polynomial. For $n \in \mathbb{Z}^+$, define

$$f_1(x) = f(x) \quad \text{and} \quad f_{n+1}(x) = f_n(f(x)) \quad \text{for } n \geq 1.$$

Let $n \in \mathbb{Z}^+$. Let $K_n \subseteq \overline{F}$ be a splitting field of $f_n(x)$ over F. Show that $K_n \subseteq K_{n+1}$.

Hint: Prove that for each root α of $f_n(x)$ there exists a root β of $f_{n+1}(x)$ such that $\alpha = f(\beta)$. For this it may be useful to write $f_{n+1}(x) = f_n(f(x))$ as a product of quadratic polynomials with coefficients in K_n.

(3) Suppose F is a field of characteristic $p > 0$. Let E/F be a finite extension and suppose $[E : F]$ is relatively prime to p. Show that E is separable over F.

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