Auslander's formula in dualizing variaties

Shijie Zhu (Joint with Ron Gentle, Job Rachowicz and Gordana Todorov)

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The Auslander's formula

Theorem (Auslander)

Let Λ be an artin algebra.

 $(\Lambda - mod) - mod$: the category of finitely presented (contravariant) functors,

 $(\Lambda - mod) - mod_0$: the category of finitely presented functors vanishing on projective modules.

Then

$$\frac{(\Lambda - \operatorname{mod}) - \operatorname{mod}}{(\Lambda - \operatorname{mod}) - \operatorname{mod}_0} \cong \Lambda - \operatorname{mod}$$

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Remark

$$(\Lambda - \text{mod}) - \text{mod}_0 = \{F | (-, X) \stackrel{(-, f)}{\rightarrow} (-, Y) \rightarrow F \rightarrow 0$$

for some epimorphism $f : X \rightarrow Y\}$
 $\cong (\underline{\Lambda - \text{mod}}) - \text{mod}$

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Let \mathcal{A} be an additive category with pseudo-kernel. i.e. for any morphism $f : \mathcal{A} \to \mathcal{B}$, there is a morphism $g : \mathcal{K} \to \mathcal{A}$ such that

$$\mathsf{Hom}(-, \mathcal{K}) \stackrel{\mathsf{Hom}(-,g)}{
ightarrow} \mathsf{Hom}(-, \mathcal{A}) \stackrel{\mathsf{Hom}(-,f)}{
ightarrow} \mathsf{Hom}(-, \mathcal{B})$$

is exact.

For example, triangulated categories have pseudo-kernels.

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Proposition

An additive category \mathcal{A} has pseudo-kernel if and only if $\mathcal{A}-mod$ is abelian.

Let \mathcal{A} be an additive category with pseudo-kernel. i.e. for any morphism $f : \mathcal{A} \to \mathcal{B}$, there is a morphism $g : \mathcal{K} \to \mathcal{A}$ such that

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For example, triangulated categories have pseudo-kernels.

Proposition

An additive category \mathcal{A} has pseudo-kernel if and only if $\mathcal{A}-mod$ is abelian.

If \mathcal{A} has pseudo-kernel, then any contravariantly finite subcategory \mathcal{X} has pseudo-kernel.

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Let \mathcal{X} be a contravariantly finite subccategory of \mathcal{A} ; \mathcal{A} -mod be the category of finitely presented functors on \mathcal{A} ; $\mathcal{T}_{\mathcal{X}} = \{F \in \mathcal{A} - \text{mod} | (-, X) \rightarrow F \rightarrow 0 \text{ for some } X \in \mathcal{X}\};$ $\mathcal{F}_{\mathcal{X}} = \{F \in \mathcal{A} - \text{mod} | F(X) = 0 \text{ for all } X \in \mathcal{X}\}.$

Definition

 $(\mathcal{F}, \mathcal{T})$ is a torsion theory in abelian category \mathcal{C} , if (1) $\mathcal{T}^{\perp} = \mathcal{F}$ and $^{\perp} \mathcal{F} = \mathcal{T}$. (2) For any $M \in \mathcal{C}$, there is an exact sequence $0 \rightarrow tM \rightarrow M \rightarrow rM \rightarrow 0$, where $tM \in \mathcal{T}$ and $rM \in \mathcal{F}$.

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Theorem (Gentle, Todorov)

 $(\mathcal{F}_{\mathcal{X}}, \mathcal{T}_{\mathcal{X}})$ is a torsion theory.

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Let $F \in \mathcal{A}-mod$.

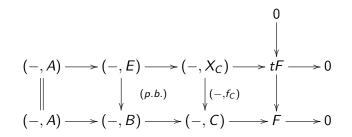
$(-, A) \longrightarrow (-, B) \longrightarrow (-, C) \longrightarrow F \longrightarrow 0$

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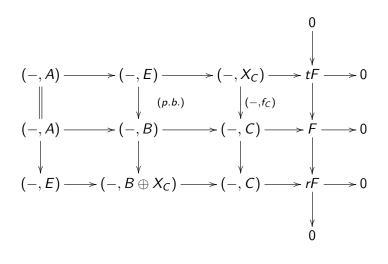
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Let $F \in \mathcal{A} - mod$.



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Denote by $res_{\mathcal{X}}$ the restriction functor.

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Denote by $\operatorname{res}_{\mathcal{X}}$ the restriction functor. Define a functor $e : \mathcal{X} - \operatorname{mod} \to \mathcal{A} - \operatorname{mod}$: If $F \in \mathcal{X}$ has a presentation

$$(\mathcal{X}, X_1) \stackrel{(\mathcal{X}, f)}{\rightarrow} (\mathcal{X}, X_0) \rightarrow F \rightarrow 0,$$

define *eF* by

$$(\mathcal{A}, X_1) \stackrel{(\mathcal{A}, f)}{\rightarrow} (\mathcal{A}, X_0) \rightarrow eF \rightarrow 0.$$

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define *eF* by

$$(\mathcal{A}, X_1) \stackrel{(\mathcal{A}, f)}{\rightarrow} (\mathcal{A}, X_0) \rightarrow eF \rightarrow 0.$$

Theorem

For any
$$F \in \mathcal{A} - \text{mod}$$
, $\text{res}_{\mathcal{X}} F \in \mathcal{X} - \text{mod}$.

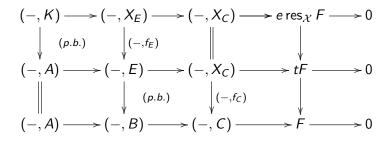
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Let $F \in \mathcal{A} - mod$.



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• $r : \mathcal{A} - \text{mod} \to \mathcal{F}_{\mathcal{X}}$ is a left adjoint of the inclusion $i : \mathcal{F}_{\mathcal{X}} \to \mathcal{A} - \text{mod}.$

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- $r : \mathcal{A} \text{mod} \to \mathcal{F}_{\mathcal{X}}$ is a left adjoint of the inclusion $i : \mathcal{F}_{\mathcal{X}} \to \mathcal{A} \text{mod}.$
- $\bullet \mbox{ res}_{\mathcal{X}}: \mathcal{A} \mbox{ mod } \rightarrow \mathcal{X} \mbox{ mod is a right adjoint of }$
- $e: \mathcal{X}-\mathsf{mod} \to \mathcal{A}-\mathsf{mod}.$

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Image: Image:

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Proposition

There is an exact sequence of categories

$$\mathcal{O} \longrightarrow \mathcal{F}_{\mathcal{X}} \xrightarrow{i} \mathcal{A} - \operatorname{mod} \xrightarrow{\operatorname{res}_{\mathcal{X}}} \mathcal{X} - \operatorname{mod} \longrightarrow \mathcal{O}$$

where *i* is the inclusion functor with $r \dashv i$ and $e \dashv \operatorname{res}_{\mathcal{X}}$.

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Question: when does the functor $res_{\mathcal{X}}$ has a right adjoint?

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Question: when does the functor $res_{\mathcal{X}}$ has a right adjoint?

Theorem (Asadollahi, J., Hafezi, R., Keshavarz, M.H, 2017)

When \mathcal{A} is a contravariantly finite subcategory of Λ -mod for some artin algebra Λ containing all the projective Λ modules and \mathcal{X} is the category of projective Λ modules, there is a recollement

$$\mathcal{F}_{\mathcal{X}} \xrightarrow{i} \mathcal{A} - \operatorname{mod} \xrightarrow{\operatorname{res}_{\mathcal{X}}} \mathcal{X} - \operatorname{mod},$$

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$$\mathcal{F}_{\mathcal{X}} \xrightarrow{i} \mathcal{A} - \operatorname{mod} \xrightarrow{\operatorname{res}_{\mathcal{X}}} \mathcal{X} - \operatorname{mod},$$

Notice in this situation, $\mathcal{F}_{\mathcal{X}} \cong \mathcal{A} - \text{mod}_0$ and $\mathcal{X} - \text{mod} \cong \Lambda - \text{mod}$.

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Definition

Let **k** be a commutative artin ring with radical **r** and $E(\mathbf{k}/\mathbf{r})$ be the injective envelope of the **k** module \mathbf{k}/r . Denote by $D = \text{Hom}_{\mathbf{k}}(-, E(\mathbf{k}/r))$ the duality. Then a Hom-finite additive **k** category C is called a dualizing **k**-variety if there is an equivalence

$$\mathcal{C}-\operatorname{mod} \rightarrow \mathcal{C}^{op}-\operatorname{mod}$$

 $F \mapsto DF.$

For example, Λ – mod is a dualizing variety. Any functorially finite subcategory of a dualizing variety is again a dualizing variety.

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Theorem (Ogawa, 2017)

When A is a dualizing variety and $X \subseteq A$ is a functorially finite subcategory, there is a recollement

$$\mathcal{F}_{\mathcal{X}} \xrightarrow{i} \mathcal{A} - \operatorname{mod} \xrightarrow{i} \mathcal{X} - \operatorname{mod}$$

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Theorem

Let C be a dualizing **k**-variety. Let A be a contravariantly finite subcategory of C and $X \subseteq A$ be a functorially finite subcategory of C. Then we have a recollement of abelian categories:

$$\mathcal{F}_{\mathcal{X}} \xrightarrow[\operatorname{coind}_{\mathcal{A}}]{\overset{i}{\longrightarrow}} \mathcal{A} - \operatorname{mod} \xrightarrow[\operatorname{coind}_{\mathcal{X}}]{\overset{i}{\longrightarrow}} \mathcal{X} - \operatorname{mod}.$$

This unifies the previous theorems.

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The right adjoint of $\operatorname{res}_{\mathcal{X}}$ is given by the coinduction functor: $\operatorname{coind}_{\mathcal{X}}F := \operatorname{Hom}(\operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, -), F).$ Since, suppose T is the right adjoint of $\operatorname{res}_{\mathcal{X}}$, then

$$coind_{\mathcal{X}}F = Hom((\mathcal{X}, -), F) = Hom(res_{\mathcal{X}}(\mathcal{A}, -), F)$$
$$= Hom((\mathcal{A}, -), TF) = TF.$$

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