Tree normal forms for quiver representations

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Conjecture [Kac]

Let $\mathrm{Ind}(Q, \alpha)$ be the set of indecomposable representations of dimension α up to isomorphism. We have

$$\mathrm{Ind}(Q, \alpha) \cong \coprod_{j \in J} \mathbb{A}^{l_j}$$

where $c_i = |\{j \in J \mid l_j = i\}|$.

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- T is a tree module if it is indecomposable and if there exists a basis $\mathcal{B}_i$ for every T_i such that the number of non-zero entries of $((T_\rho)_{\mathcal{B}_i,\mathcal{B}_j})_{\rho:i\to j\in Q_1}$ is $(\sum_{i\in Q_0} \dim T_i) - 1$.

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- For any pair of representations M, N, there exists a natural map

$$d_{M,N} : H_{M,N} := \bigoplus_{\rho:i\to j\in Q_1} \mathrm{Hom}(M_i, N_j) \twoheadrightarrow \mathrm{Ext}(M,N),$$

$$(g_\rho)_{\rho\in Q_1} \mapsto \left(0 \to (N_\rho)_{\rho\in Q_1} \to \begin{pmatrix} N_\rho & g_\rho \\ 0 & M_\rho \end{pmatrix}_{\rho\in Q_1} \to (M_\rho)_{\rho\in Q_1} \to 0\right)$$

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- We can thus choose $f_1,\dots,f_n \in H_{M,N}$ such that $(d_{M,N}(f_i))_i$ is a basis of $\mathrm{Ext}(M,N)$.

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$$T(\mu)_\rho = T_\rho + \sum_{i=1}^{n} \mu_i (f_i)_\rho$$

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- A subspace $U \subset \langle f_1, \ldots, f_n \rangle$ is called strong and separating if $T(\mu)$ is indecomposable for every $\mu \in U$ and if $T(\lambda) \ncong T(\mu)$ for $\mu, \lambda \in U$ with $\lambda \neq \mu$.

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- A dimension vector $\alpha \in \mathbb{N}^{Q_0}$ admits a (unique) tree normal form if there exist tree modules $T_1, \dots, T_r \in \mathrm{Rep}(Q, \alpha)$ and strong and separating subspaces $U_{T_i} \subset \mathrm{Ext}(T_i, T_i)$ such that for every indecomposable representation $M \in \mathrm{Rep}(Q, \alpha)$ there exists a (unique) $i \in \{1, \dots, r\}$ such that $M \cong T_i(\mu_i)$ for some $\mu_i \in U_{T_i}$.

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Theorem

Let $S, \hat{T}$ be tree modules with strong and separating subspaces $U_S,\ U_{\hat{T}}$ such that $\mathrm{End}(S(\mu)) = k$, $\mathrm{Hom}(\hat{T}(\lambda), S(\mu)) = \mathrm{Hom}(S(\mu), \hat{T}(\lambda)) = 0$ for all $\mu \in U_S, \lambda \in U_{\hat{T}}$. Let $e \in \mathrm{Ext}(S, \hat{T})$ be universal for U_S and $U_{\hat{T}}$. Then $U_S + U_{\hat{T}} \subseteq \mathrm{Ext}(T, T)$ is strong and separating where T is the middle term of $e : 0 \to \hat{T} \to T \to S \to 0$.

Thank you!