Partition Identities and Quiver Representations

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November 20th, 2017

Based on joint work with Richárd Rimányi and Alexander Yong

arXiv:1608.02030

Overview

This project provides an elementary explanation for a **quantum dilogarithm identity** due to M. Reineke.

We use **generating function** techniques to establish a related identity, which is a generalization of the **Euler-Gauss** identity.

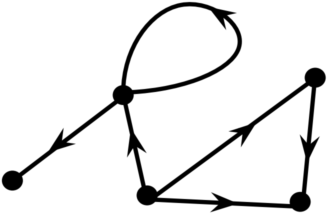
This reduces to an equivalent form of Reineke's identity in type A.

Representations of Quivers

Quivers

A **quiver** $Q = (Q_0, Q_1)$ is a directed graph with

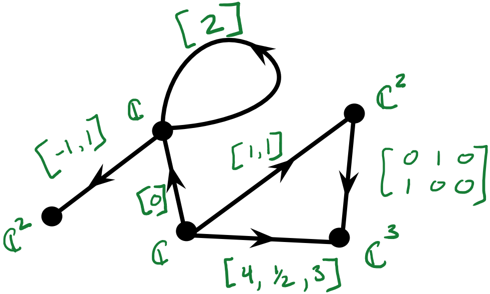
- **vertices**: $i \in Q_0$
- **edges**: $a : i \to j \in Q_1$



The Definition

A **representation** of Q is an assignment of a:

- **vector space** V_i to each vertex $i \in Q_0$ and
- **linear transformation** $f_a : V_i \to V_j$ to each arrow $i \xrightarrow{a} j \in Q_1$



$\mathbf{dim}(V) = (\mathtt{dim} V_i)_{i \in Q_0}$ is the **dimension vector** of V.

The Representation Space

Fix $\mathbf{d} \in \mathbb{N}^{Q_0}$. The **representation space** is

$$\mathrm{Rep}_Q(\mathbf{d}) := \bigoplus_{i \xrightarrow{a} j \in Q_1} \mathrm{Mat}(\mathbf{d}(i), \mathbf{d}(j)).$$

Let

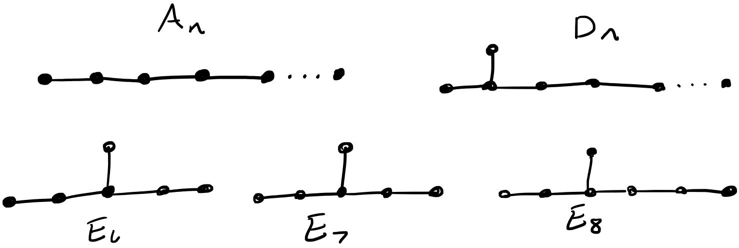
$$\mathrm{GL}_Q(\mathbf{d}) := \prod_{i \in Q_0} \mathrm{GL}(\mathbf{d}(i)).$$

$\mathrm{GL}_Q(\mathbf{d})$ acts on $\mathrm{Rep}_Q(\mathbf{d})$ by base change at each vertex.

Orbits of this action are in bijection with isomorphism classes of $\mathbf{d}$ dimensional representations.

Dynkin Quivers

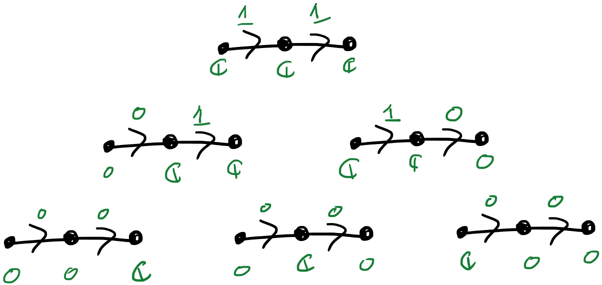
A quiver is **Dynkin** if its underlying graph is of type *ADE*:



Gabriel's Theorem

Theorem ([Gab75])

Dynkin *quivers have finitely many isomorphism classes of indecomposable representations.*

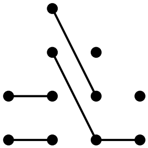


For type A, indecomposables $V_{[i,j]}$ are indexed by **intervals**.

Lacing Diagrams

A **lacing diagram** ([ADF85]) $\mathcal{L}$ is a graph so that:

- the vertices are arranged in n columns labeled $1, 2, \ldots, n$
- the edges between adjacent columns form a partial matching.



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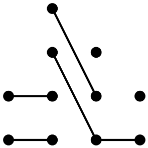
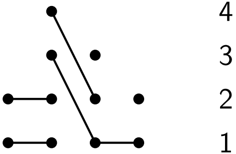


Idea: Lacing diagrams are a way to visually encode representations of an A_n quiver.

The Role of Lacing Diagrams in Representation Theory

When Q is a type A quiver, a lacing diagram can be interpreted as a sequence of **partial permutation matrices** which form a representation $V_{\mathcal{L}}$ of Q.

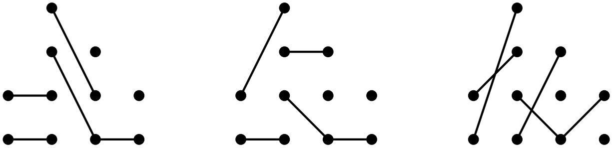


$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

See [KMS06] for the **equiorientated case** and [BR04] for **arbitrary orientations**.

Equivalence Classes of Lacing Diagrams

Two lacing diagrams are **equivalent** if one can be obtained from the other by permuting vertices within a column.



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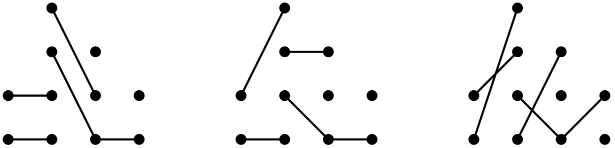
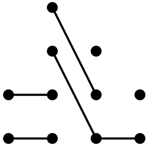


$$\{\text{Equivalence Classes of Lacing Diagrams}\}$$
$$\updownarrow$$
$$\{\text{Isomorphism Classes of Representations of } Q\}$$

Strands

A **strand** is a connected component of $\mathcal{L}$.



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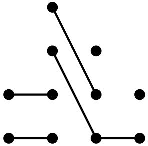

$m_{[i,j]}(\mathcal{L}) = |\{\text{strands starting at column } i \text{ and ending at column } j\}|$

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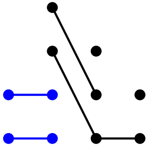

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Example: $m_{[1,2]}(\mathcal{L}) = 2$

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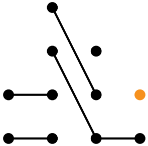


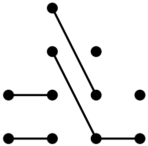
$m_{[i,j]}(\mathcal{L}) = |\{\text{strands starting at column } i \text{ and ending at column } j\}|$

Example: $m_{[4,4]}(\mathcal{L}) = 1$

Strands

Strands record the decomposition of $V_{\mathcal{L}}$ into indecomposable representations:

$$V_{\mathcal{L}} \cong \bigoplus V_{[i,j]}^{\oplus m_{[i,j]}(\mathcal{L})}$$



Example: $V_{\mathcal{L}} \cong V_{[1,2]}^{\oplus 2} \oplus V_{[2,3]} \oplus V_{[2,4]} \oplus V_{[3,3]} \oplus V_{[4,4]}$

Reineke's Identities

The Quantum Dilogarithm Series

$$\mathbb{E}(z) = \sum_{k=0}^{\infty} \frac{q^{k^2/2} z^k}{(1-q)(1-q^2)\dots(1-q^k)}$$

The Quantum Algebra of a Quiver

The **Quantum Algebra** $\mathbb{A}_Q$ is an algebra over $\mathbb{Q}(q^{1/2})$ with

- generators:

$$\{y_{\mathbf{d}} : \mathbf{d} \in \mathbb{N}^{Q_0}\}$$

- multiplication:

$$y_{\mathbf{d}_1} y_{\mathbf{d}_2} = q^{\frac{1}{2}(\chi(\mathbf{d}_2,\mathbf{d}_1)-\chi(\mathbf{d}_1,\mathbf{d}_2))} y_{\mathbf{d}_1+\mathbf{d}_2}$$

The **Euler form** $\chi : \mathbb{N}^{Q_0} \times \mathbb{N}^{Q_0} \to \mathbb{Z}$

$$\chi(\mathbf{d}_1, \mathbf{d}_2) = \sum_{i \in Q_0} \mathbf{d}_1(i)\mathbf{d}_2(i) \quad - \sum_{i \xrightarrow{a} j \in Q_1} \mathbf{d}_1(i)\mathbf{d}_2(j)$$

Reineke's Identity

Given a representation V, we'll write $\mathbf{d}_V$ as a shorthand for $\mathbf{d}_{\mathtt{dim}(V)}$.

For a Dynkin quiver, it is possible to fix a choice of ordering on the

- simple representations: $\alpha_1, \ldots, \alpha_n$
- indecomposable representations: $\beta_1, \ldots, \beta_N$

so that

$$\mathbb{E}(y_{\mathbf{d}_{\alpha_1}}) \cdots \mathbb{E}(y_{\mathbf{d}_{\alpha_n}}) = \mathbb{E}(y_{\mathbf{d}_{\beta_1}}) \cdots \mathbb{E}(y_{\mathbf{d}_{\beta_N}}) \qquad (1)$$

(Original proof given by [Rei10], see [Kel11] for exposition and a sketch of the proof.)

Reformulation

Looking at the coefficient of $y_{\mathbf{d}}$ on each side, this is equivalent to the following infinite family of identities ([Rim13]):

$$\prod_{i=1}^{n} \frac{1}{(q)_{\mathbf{d}(i)}} = \sum_{\eta} q^{\mathtt{codim}_{\mathbb{C}}(\eta)} \prod_{i=1}^{N} \frac{1}{(q)_{m_{\beta_i}(\eta)}}.$$

where the sum is over orbits η in $\mathrm{Rep}_Q(\mathbf{d})$ and $m_\beta(\eta)$ is the multiplicity of β in $V \in \eta$.

Here, $(q)_k = (1-q)\cdots(1-q^k)$ is the q-shifted factorial.

Some Bookkeeping

Fix a sequence of permutations $\mathbf{w} = (w^{(1)}, \ldots, w^{(n)})$, so that $w^{(i)} \in \mathfrak{S}_i$ and $w^{(i)}(i) = i$.

Let

$$s_i^j(\mathcal{L}) = m_{[i,j-1]}(\mathcal{L})$$

and

$$t_i^k(\mathcal{L}) = m_{[i,k]}(\mathcal{L}) + m_{[i,k+1]}(\mathcal{L}) + \ldots + m_{[i,n]}(\mathcal{L}).$$

Define the **Durfee statistic**:

$$r_{\mathbf{w}}(\mathcal{L}) = \sum_{1 \leq i < j \leq k \leq n} s^k_{w^{(k)}(i)}(\mathcal{L}) t^k_{w^{(k)}(j)}(\mathcal{L}).$$

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The above statistics are all constant on equivalence classes of lacing diagrams.

Theorem (Rimányi, Weigandt, Yong, 2016)

Fix a dimension vector $\mathbf{d} = (\mathbf{d}(1), \ldots, \mathbf{d}(n))$ *and let* $\mathbf{w}$ *be as before. Then*

$$\prod_{k=1}^{n} \frac{1}{(q)_{\mathbf{d}(k)}} = \sum_{\eta \in \mathrm{L}(\mathbf{d})} q^{r_{\mathbf{w}}(\eta)} \prod_{k=1}^{n} \frac{1}{(q)_{t_k^k(\eta)}} \prod_{i=1}^{k-1} \begin{bmatrix} t_i^k(\eta) + s_i^k(\eta) \\ s_i^k(\eta) \end{bmatrix}_q$$

$\begin{bmatrix} j+k \\ k \end{bmatrix}_q$ is the q-**binomial coefficient** and $(q)_k$ the q-**shifted factorial**.

Generating Series for Partitions

Generating Series

Let S be a set equipped with a weight function

$$\mathtt{wt} : S \to \mathbb{N}$$

so that

$$|\{s \in S : \mathtt{wt}(s) = k\}| < \infty$$

for each $k \in \mathbb{N}$.

The **generating series** for S is

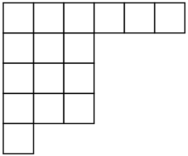
$$G(S, q) = \sum_{s \in S} q^{\mathtt{wt}(s)}.$$

Partitions

An **integer partition** is an ordered list of decreasing integers:

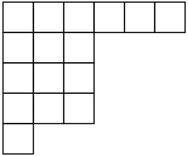
$$\lambda = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell(\lambda)} > 0$$

We will typically represent a partition by its **Young diagram**:



We weight a partition by counting the boxes in its Young diagram.

Generating Series for Partitions



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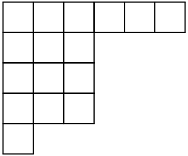

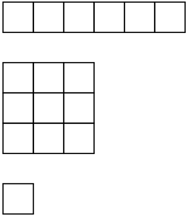

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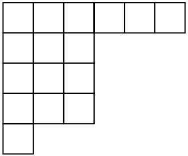

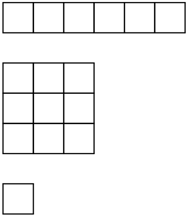

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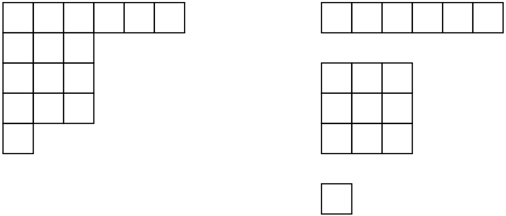

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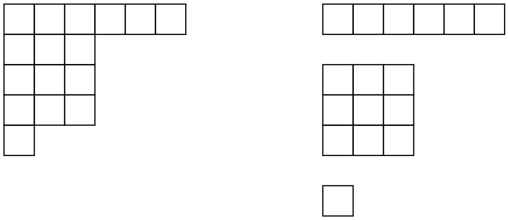

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$$q^{16} = q^1 \cdot 1 \cdot q^{3\cdot 3} \cdot 1 \cdot 1 \cdot q^6 \cdot 1 \cdot 1 \cdot \ldots$$

Notation

- Let $\mathcal{R}(j,k)$ be the set consisting of a **single rectangular partition** of size $j \times k$.

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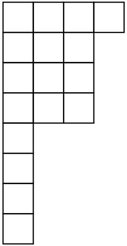
$$G(\mathcal{P}(j, k), q) = ??$$

$\mathcal{P}(\infty, k)$: Partitions With at Most k Columns

Idea: Truncate the product



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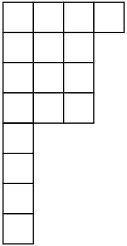
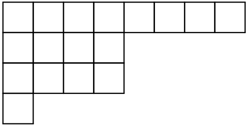
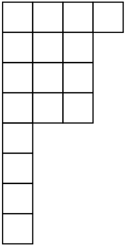


$$\frac{1}{(q)_k} = \prod_{i=1}^{k} \frac{1}{(1-q^i)}$$

$\mathcal{P}(k, \infty)$: Partitions With at Most k Rows

Idea: Bijection via conjugation





$\mathcal{P}(k,\infty)$: Partitions With at Most k Rows

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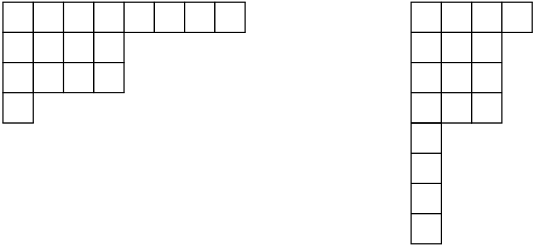


$$\frac{1}{(q)_k} = \prod_{i=1}^{k} \frac{1}{(1-q^i)}$$

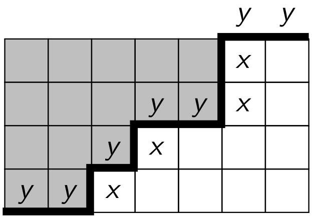
q-Binomial Coefficients

q-**Binomial Coefficient**:
If x and y are q commuting ($yx = qxy$),

$$(x+y)^n = \sum_{i+j=n} \begin{bmatrix} i+j \\ i \end{bmatrix}_q x^i y^j.$$

Combinatorial Interpretation of the q-Binomial Coefficient

$$(x+y)^{11} = \ldots + yyxyxyyxxyy + \ldots$$



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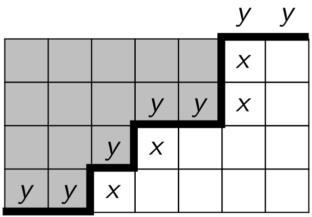

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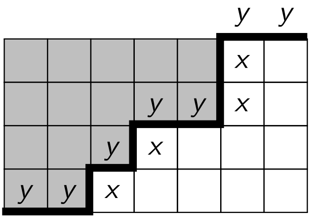
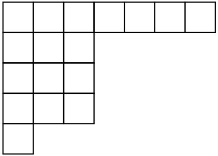


$$G(\mathcal{P}(i,j),q) = \begin{bmatrix} i+j \\ i \end{bmatrix}_q = \frac{(q)_{i+j}}{(q)_i (q)_j}$$

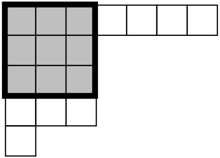
Durfee Squares and Rectangles

Durfee Squares



The **Durfee square** $D(\lambda)$ is the largest $j \times j$ square partition contained in λ.

Durfee Squares



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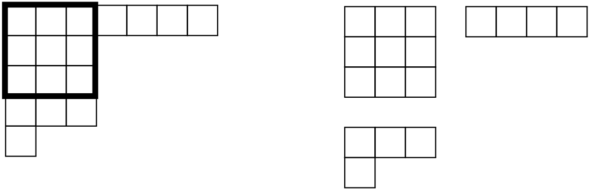
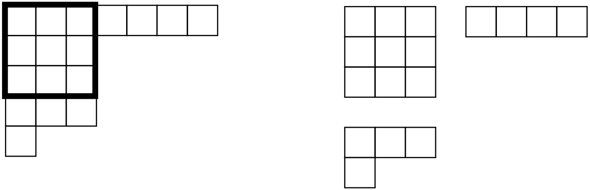


$$\mathcal{P}(\infty,\infty) \quad \longleftrightarrow \quad \bigcup_{j\geq 0} \mathcal{R}(j,j) \times \mathcal{P}(j,\infty) \times \mathcal{P}(\infty,j)$$

Durfee Squares



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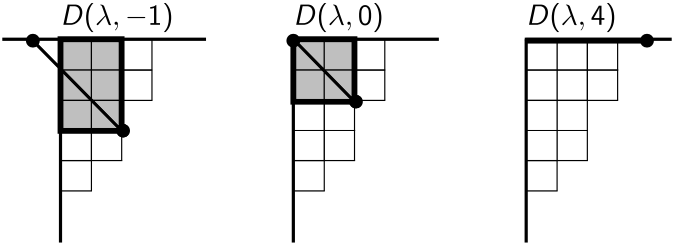
Euler-Gauss identity:

$$\frac{1}{(q)_\infty} = \sum_{j=0}^{\infty} \frac{q^{j^2}}{(q)_j (q)_j} \tag{2}$$

See [And98] for details and related identities.

Durfee Rectangles



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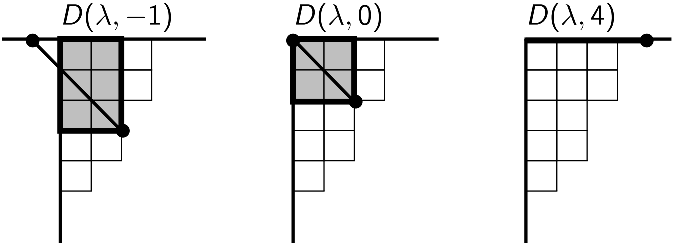
Durfee Rectangles



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$$\frac{1}{(q)_\infty} = \sum_{s=0}^{\infty} \frac{q^{s(s+r)}}{(q)_s (q)_{s+r}}. \tag{3}$$

([GH68])

Proof of the Main Theorem

Theorem (Rimányi, Weigandt, Yong, 2016)

Fix a dimension vector $\mathbf{d} = (\mathbf{d}(1), \ldots, \mathbf{d}(n))$ *and let* $\mathbf{w}$ *be as before. Then*

$$\prod_{k=1}^{n} \frac{1}{(q)_{\mathbf{d}(k)}} = \sum_{\eta \in \mathrm{L}(\mathbf{d})} q^{r_{\mathbf{w}}(\eta)} \prod_{k=1}^{n} \frac{1}{(q)_{t_k^k(\eta)}} \prod_{i=1}^{k-1} \begin{bmatrix} t_i^k(\eta) + s_i^k(\eta) \\ s_i^k(\eta) \end{bmatrix}_q$$

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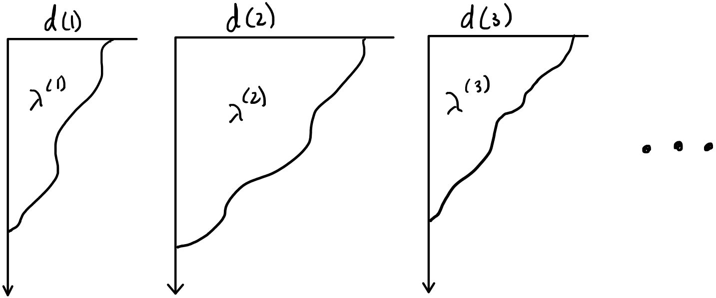
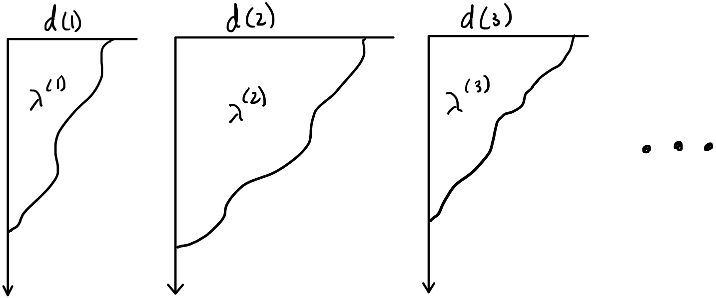
Idea: We will interpret each side as a generating series for tuples of partitions. Giving a weight preserving bijection between these two sets proves the identity.

$$S = \mathcal{P}(\infty, \mathbf{d}(1)) \times \ldots \times \mathcal{P}(\infty, \mathbf{d}(n))$$



The Left Hand Side

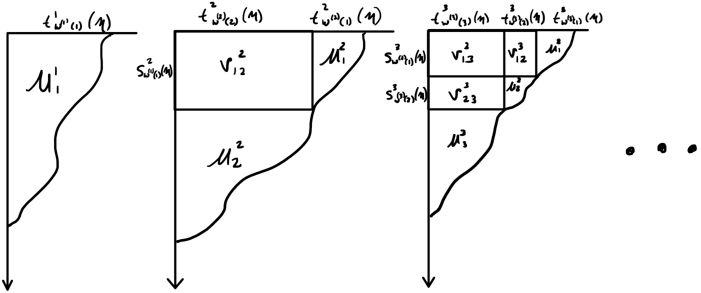
$$S = \mathcal{P}(\infty, \mathbf{d}(1)) \times \ldots \times \mathcal{P}(\infty, \mathbf{d}(n))$$



$$G(S, q) = \prod_{i=1}^{n} G(\mathcal{P}(\infty, \mathbf{d}(i)), q) = \prod_{i=1}^{n} \frac{1}{(q)_{\mathbf{d}(i)}}$$

The Right Hand Side

$$T = \bigcup_{\eta \in \mathrm{L}(\mathbf{d})} R(\eta) \times P(\eta)$$



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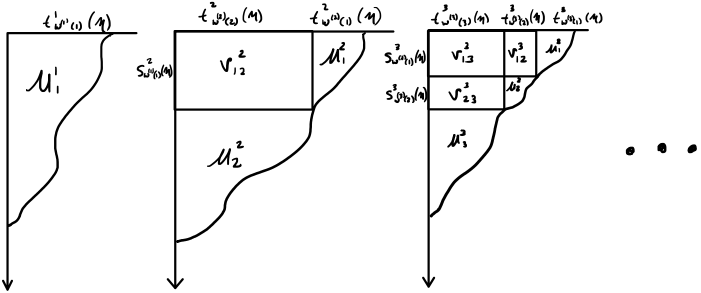
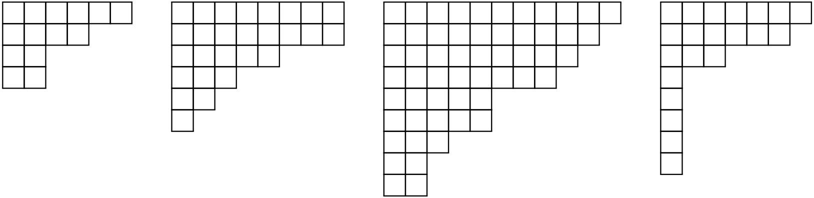


$$G(T, q) = \sum_{\eta \in \mathrm{L}(\mathbf{d})} G(R(\eta), q) G(P(\eta), q)$$

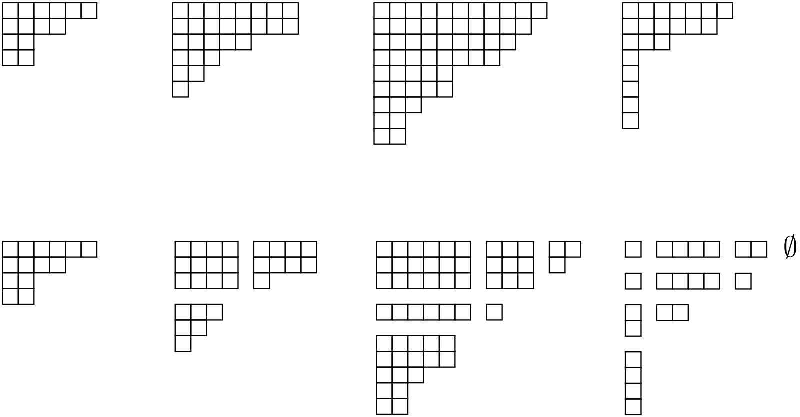
The map $S \to T$

Let $\mathbf{w} = (1, 12, 123, 1234)$ and $\mathbf{d} = (8, 9, 11, 8)$.

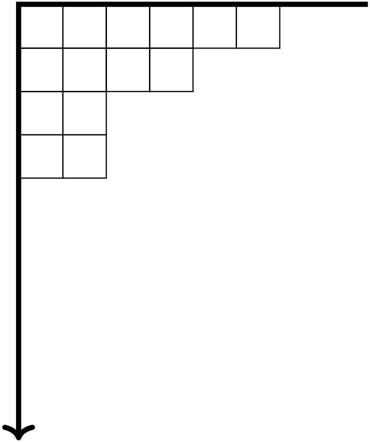


$$\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)}) \in S$$

$\boldsymbol{\lambda} \mapsto (\boldsymbol{\mu}, \boldsymbol{\nu})$



$\lambda^{(1)}$



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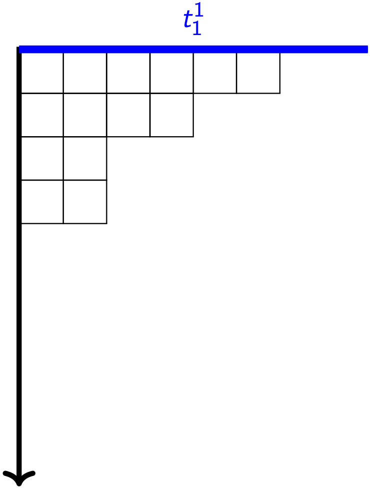
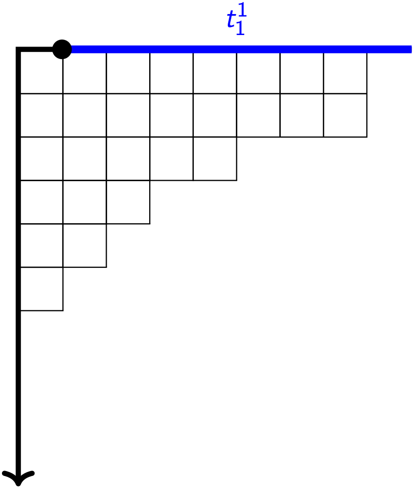
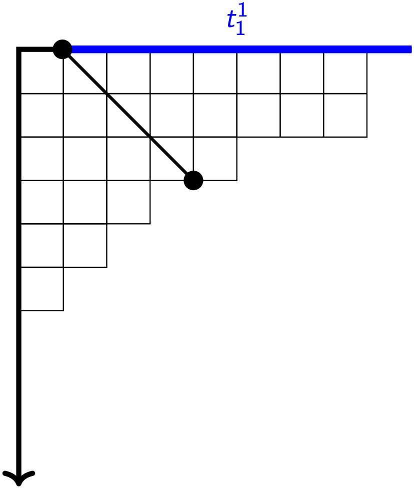


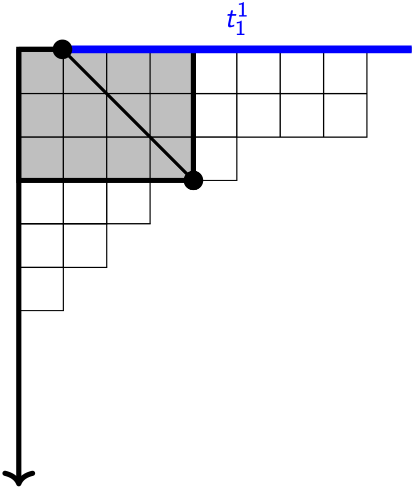
$\lambda^{(2)}$



$\lambda^{(2)}$



$\lambda^{(2)}$



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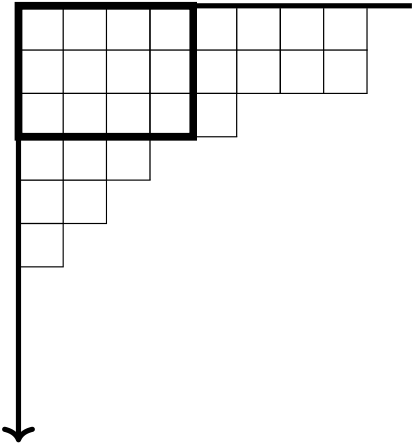
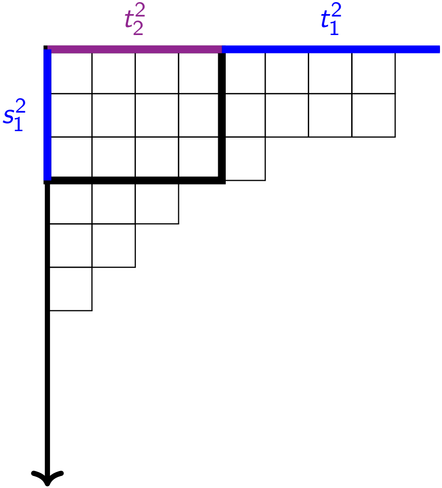
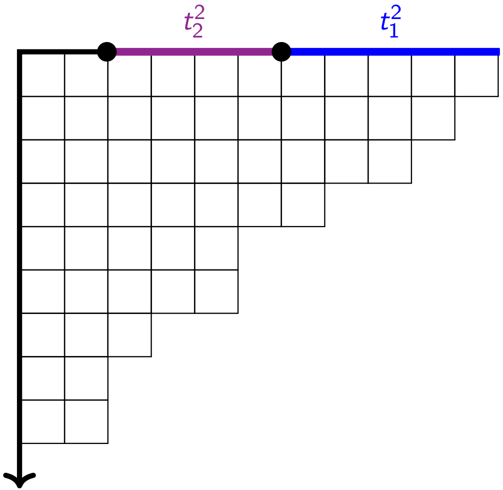


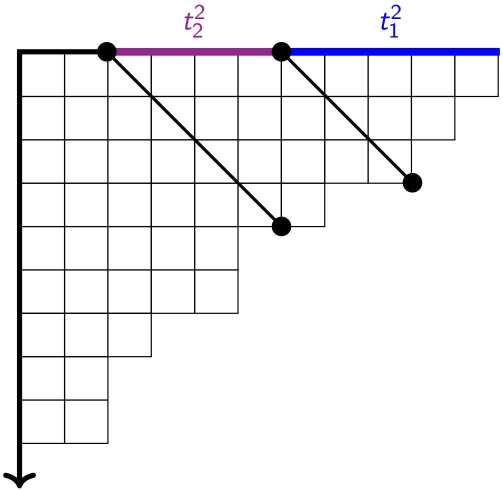
$\lambda^{(2)}$



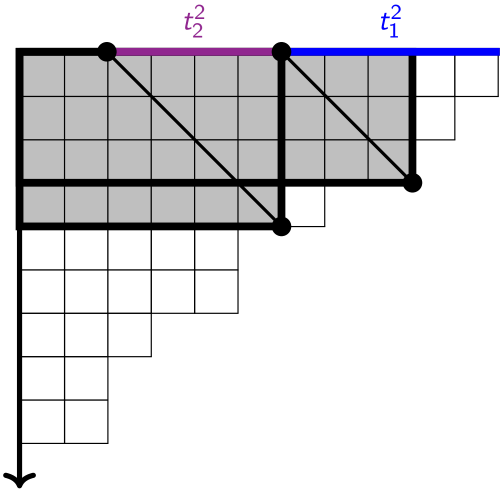
$\lambda^{(3)}$



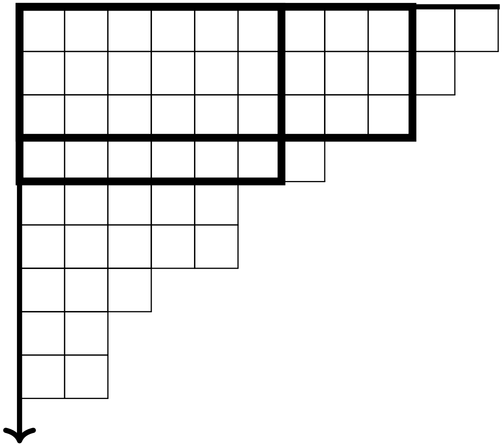
$\lambda^{(3)}$



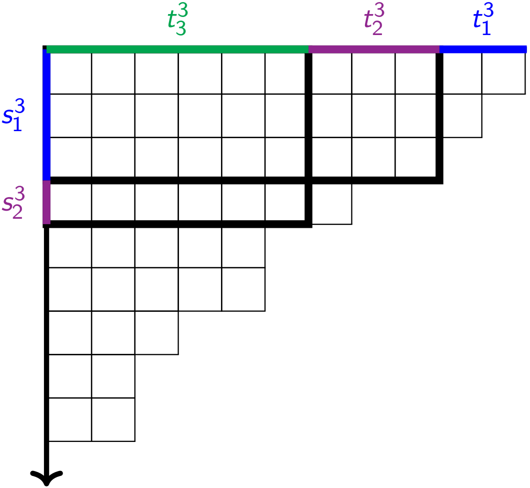
$\lambda^{(3)}$



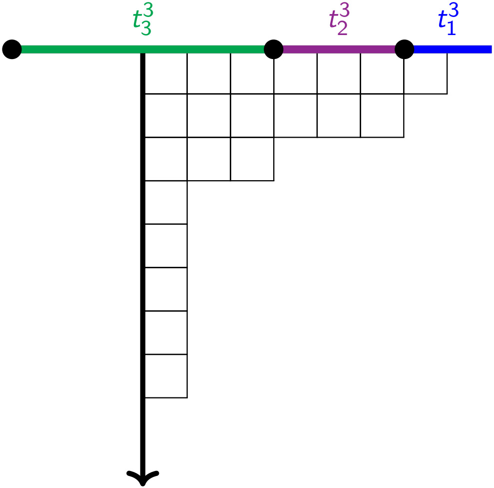
$\lambda^{(3)}$



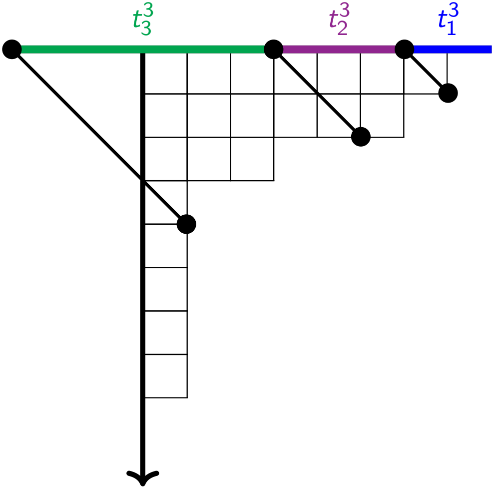
$\lambda^{(3)}$



$\lambda^{(4)}$



$\lambda^{(4)}$



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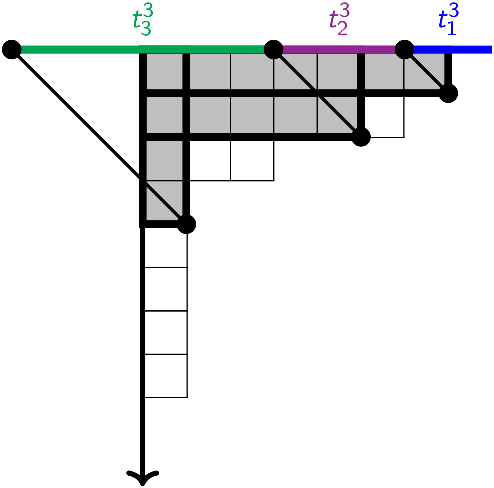

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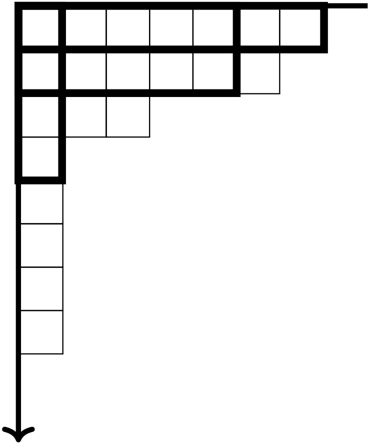
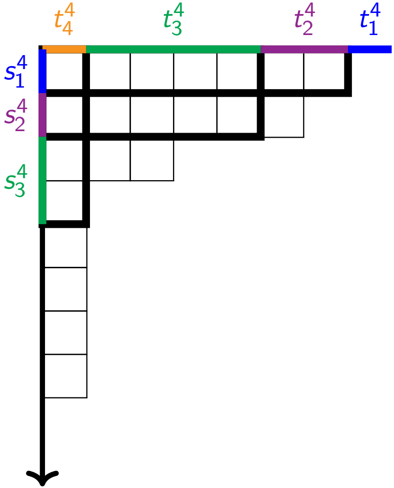


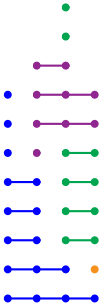
$\lambda^{(4)}$



These Parameters are Well Defined

Lemma

There exists a unique $\eta \in \mathtt{L}(\mathbf{d})$ *so that* $s_i^k(\eta) = s_i^k$ *and* $t_j^k(\eta) = t_j^k$ *for all* i, j, k.



A Recursion

For any η,

$$s_i^k(\eta) + t_i^k(\eta) = t_i^{k-1}(\eta) \tag{4}$$

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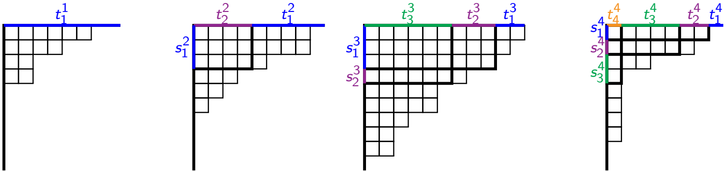
The parameters defined by Durfee rectangles satisfy the *same* equations:



$$\boldsymbol{\lambda} \mapsto (\boldsymbol{\mu}, \boldsymbol{\nu}) \in T(\eta) \subseteq T$$

Connection with Reineke's Identity (in type A)

Simplifying the Identity

Lets think about the blue terms:

$$\frac{1}{(q)_{t_1^1}} \begin{bmatrix} s_1^2 + t_1^2 \\ s_1^2 \end{bmatrix}_q \begin{bmatrix} s_1^3 + t_1^3 \\ s_1^3 \end{bmatrix}_q \begin{bmatrix} s_1^4 + t_1^4 \\ s_1^4 \end{bmatrix}_q$$

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$$= \left(\frac{1}{(q)_{t_1^1}} \right) \left(\frac{(q)_{s_1^2+t_1^2}}{(q)_{s_1^2}(q)_{t_1^2}} \right) \left(\frac{(q)_{s_1^3+t_1^3}}{(q)_{s_1^3}(q)_{t_1^3}} \right) \left(\frac{(q)_{s_1^4+t_1^4}}{(q)_{s_1^4}(q)_{t_1^4}} \right)$$

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$$= \left(\frac{1}{(q)_{t_1^1}}\right)\left(\frac{(q)_{s_1^2+t_1^2}}{(q)_{s_1^2}(q)_{t_1^2}}\right)\left(\frac{(q)_{s_1^3+t_1^3}}{(q)_{s_1^3}(q)_{t_1^3}}\right)\left(\frac{(q)_{s_1^4+t_1^4}}{(q)_{s_1^4}(q)_{t_1^4}}\right)$$

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$$= \frac{1}{(q)_{s_1^2}(q)_{s_1^3}(q)_{s_1^4}(q)_{t_1^4}}$$

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$$= \left(\frac{1}{(q)_{t_1^1}}\right)\left(\frac{(q)_{s_1^2+t_1^2}}{(q)_{s_1^2}(q)_{t_1^2}}\right)\left(\frac{(q)_{s_1^3+t_1^3}}{(q)_{s_1^3}(q)_{t_1^3}}\right)\left(\frac{(q)_{s_1^4+t_1^4}}{(q)_{s_1^4}(q)_{t_1^4}}\right)$$

$$= \frac{1}{(q)_{s_1^2}(q)_{s_1^3}(q)_{s_1^4}(q)_{t_1^4}}$$

$$= \frac{1}{(q)_{m_{[1,1]}}(q)_{m_{[1,2]}}(q)_{m_{[1,3]}}(q)_{m_{[1,4]}}}$$

Simplifying the Identity

Doing these cancellations yields the identity:

Corollary (Rimányi, Weigandt, Yong, 2016)

$$\prod_{i=1}^{n} \frac{1}{(q)_{\mathbf{d}(i)}} = \sum_{\eta \in \mathrm{L}(\mathbf{d})} q^{r_{\mathbf{w}}(\eta)} \prod_{1 \leq i \leq j \leq n} \frac{1}{(q)_{m_{[i,j]}(\eta)}}.$$

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which looks very similar to:

$$\prod_{i=1}^{n} \frac{1}{(q)_{\mathbf{d}(i)}} = \sum_{\eta} q^{\mathtt{codim}_{\mathbb{C}}(\eta)} \prod_{i=1}^{N} \frac{1}{(q)_{m_{\beta_i}(\eta)}}.$$

A Special Sequence of Permutations

We associate permutations $w_Q^{(i)} \in \mathfrak{S}_i$ to Q as follows:

- Let $w_Q^{(1)} = 1$ and $w_Q^{(2)} = 12$.
- If a_{i-2} and a_{i-1} point in the same direction, append i to $w_Q^{(i-1)}$
- If a_{i-2} and a_{i-1} point in opposite directions, reverse $w_Q^{(i-1)}$ and then append i

$$\mathbf{w}_Q := (w_Q^{(1)}, \ldots, w_Q^{(n)})$$

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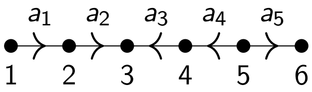
Example:



- $\mathbf{w}_Q = (1, 12, 123, 3214, 32145, 541236)$

The Geometric Meaning of $r_{\mathbf{w}}(\eta)$

The Durfee statistic has the following geometric meaning:

Theorem (Rimányi, Weigandt, Yong, 2016)

$$\mathrm{codim}_{\mathbb{C}}(\eta) = r_{\mathbf{w}_Q}(\eta)$$

The above statement combined with the corollary implies Reineke's quantum dilogarithm identity in type A.

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Thank You!