Versal deformation rings and symmetric special biserial algebras

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Motivation

Let k be an algebraically closed field, G be a finite group such that $char(k)$ divides the order of G, and M be a finitely generated kG-module.

- M has a versal deformation ring $R(G, M)$ - Mazur
- $R(G, M)$ is universal when $\underline{End}_{kG}(M) \cong k$ - Bleher-Chinburg
  - $R(G, M)$ classified for all M belonging to a block of kG of finite representation type where $\underline{End}_{kG}(M) \cong k$
- Initial goal: Consider M with $\underline{End}_{kG}(M) \ncong k$
  - Generalized and solved for finite representation type - Bleher-W

Generalization and goals

Let k be an algebraically closed field of arbitrary characteristic, Λ be a symmetric special biserial algebra over k, and M be a finitely generated Λ-module.

- $R(\Lambda, M)$ classified for all M when Λ of finite type - Bleher-W
- Goal: Consider Λ of domestic representation type
  - Classify $R(\Lambda, M)$ for all M where $\underline{End}_{\Lambda}(M) \cong k$ (Universal)
  - Loosen restriction and classify versal deformation rings

To do this, we study an equivalent family of algebras: Brauer graph algebras. (Schroll)

Special biserial algebras

Let k be an algebraically closed field of arbitrary characteristic.

Definition

A finite dimensional k-algebra Λ is called *special biserial* if there is a quiver Q and an admissible ideal I in kQ such that Λ is Morita equivalent to kQ/I and such that kQ/I satisfies the following conditions:

- At every vertex v in Q there are at most two arrows starting at v and at most two arrows ending at v,
- For every arrow α in Q there exists at most one arrow β such that $\alpha\beta \notin I$ and at most one arrow γ such that $\gamma\alpha \notin I$.

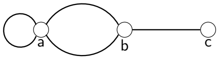
We consider the case where our special biserial algebra is also symmetric.

Brauer graphs

Let k be an algebraically closed field of arbitrary characteristic.

Definition

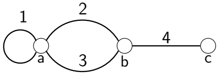
A Brauer graph G is a finite, undirected, graph together with a cyclic ordering of the edges emanating from each vertex, i, along with a positive integer value $m(i)$ assigned to each vertex.



$m(a) = m(c) = 1, m(b) = 2$

Brauer graph algebras

A finite dimensional k-algebra Λ is called a Brauer graph algebra if there is a related Brauer graph $G(\Lambda)$ which encodes all projective indecomposable Λ-modules. For example, if $G(\Lambda)$ is as follows:



$m(a) = m(c) = 1, m(b) = 2$

then the related Brauer graph algebra Λ has the following projective indecomposable modules:

$$P_1\colon \begin{array}{ccc} & 1 & \\ 2 & & 1 \\ 3 & & 2 \\ 1 & & 3 \\ & 1 & \end{array} \qquad P_2\colon \begin{array}{ccc} & 2 & \\ 3 & & 4 \\ 1 & & 3 \\ 1 & & 2 \\ & & 4 \\ & & 3 \\ & 2 & \end{array} \qquad P_3\colon \begin{array}{ccc} & 3 & \\ 1 & & 2 \\ 1 & & 4 \\ 2 & & 3 \\ & & 2 \\ & & 4 \\ & 3 & \end{array} \qquad P_4\colon \begin{array}{c} 4 \\ 3 \\ 2 \\ 4 \\ 3 \\ 2 \\ 4 \end{array}$$

The versal deformation ring of a module

Let $\hat{\mathcal{C}}$ be the category of complete local Noetherian commutative k-algebras, and $R \in Ob(\hat{\mathcal{C}})$. Let V be a finitely generated Λ-module.

Definition

A lift of V over R is an $R \otimes_k \Lambda$-module M which is free as an R-module together with a Λ-module isomorphism $\phi : k \otimes_R M \to V$.

We say V has a versal deformation ring $R(\Lambda, V)$ in $\hat{\mathcal{C}}$ if every isomorphism class of lifts of V over every $R \in Ob(\hat{\mathcal{C}})$ arises from a (not necessarily unique) k-algebra homomorphism from $R(\Lambda, V)$ to R. In addition, when $R = k[\epsilon]/(\epsilon^2)$, the k-algebra homomorphism is unique.

Properties of $R(\Lambda, V)$

Bleher-Velez proved that every finitely generated Λ-module V has a versal deformation ring.

The versal deformation ring for V is of the form $k[[t_1, \ldots, t_n]]/J$ for some ideal J, where the following properties hold:

- the k dimension of $\mathrm{Ext}^1_\Lambda(V, V)$ is the minimal number of necessary variables t_i,
- the k dimension of $\mathrm{Ext}^2_\Lambda(V, V)$ is an upper bound on the minimal number of generators for the ideal J.

Imdecomposable modules for Brauer graph algebras

For a Brauer graph algebra Λ which is m-domestic, the stable Auslander-Reiten quiver has the following components:

- m components of the form $\mathbb{Z}\tilde{A}_{p,q}$
- m components of the form $\mathbb{Z}A_\infty/(\tau^p)$
- m components of the form $\mathbb{Z}A_\infty/(\tau^q)$
- infintely many components of the form $\mathbb{Z}A_\infty/(\tau)$

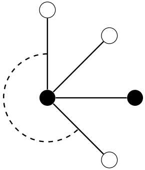
Since Λ is symmetric, the syzygy functor Ω induces an automorphism of the AR quiver, and for any non-projective indecomposable Λ-module V, $\Omega^2(V)$ is the AR-translate of V.

The versal deformation ring of V is uniquely determined by the component and "row" of the AR quiver on which V appears.

Reduction to a generalized star

Goal: Determine $R(\Lambda, V)$ for all indecomposable modules for Brauer graph algebras of domestic type.

Every Brauer graph algebra is stably equivalent (of morita type) to a Brauer graph algebra Λ with a related generalized star Brauer tree with at most 2 vertices of multiplicity greater than 1. (Kauer)



For finite type, at most one vertex has multiplicity greater than 1.
For domestic type, the 2 vertices have multiplicity 2.

Stable AR quiver for algebras of domestic type

The stable Aulander-Reiten quiver for a domestic star Brauer graph algebra with n edges has the following components:

- 1 component of the form $\mathbb{Z}\tilde{A}_{n,n}$
- 2 components of the form $\mathbb{Z}A_\infty/(\tau^n)$
- infintely many components of the form $\mathbb{Z}A_\infty/(\tau)$

The only indecomposable modules for which $\underline{End}_\Lambda(V) \cong k$ are string modules. These modules consist of:

- all modules from the component $\mathbb{Z}\tilde{A}_{n,n}$
- n rows of modules from the components $\mathbb{Z}A_\infty/(\tau^n)$

When $\underline{End}_\Lambda(V) \cong k$

Theorem: Meyer-Soto-W

Let Λ be a symmetric special biserial algebra of domestic representation type and suppose V is an indecomposable Λ-module with $\underline{End}_\Lambda(V) \cong k$. The Universal deformation ring $R(\Lambda, V)$ is isomorphic to k, $k[t]/(t^2)$, or $k[[t]]$.

$R(\Lambda, V) \cong k$ occurs when $dim_k Ext^1_\Lambda(V, V) = 0$, and these modules arise in all three components with string modules.
$R(\Lambda, V) \cong k[t]/(t^2)$ occurs when $dim_k Ext^1_\Lambda(V, V) = 1$, and these modules arise in the component of the form $\mathbb{Z}\tilde{A}_{n,n}$.
$R(\Lambda, V) \cong k[[t]]$ occurs when $dim_k Ext^1_\Lambda(V, V) = 1$, and these modules arise in the components of the form $\mathbb{Z}A_\infty/(\tau^n)$.

When $\underline{End}_\Lambda(V) \ncong k$

Conjecture: Meyer-Soto-W

Let Λ be a symmetric special biserial algebra of domestic representation type and suppose V is an indecomposable Λ-module with $\underline{End}_\Lambda(V) \ncong k$.

- If V is a string module and $dim_k Ext^1_\Lambda(V, V) = d$, then $R(\Lambda, V) \cong k[[t_1, \ldots, t_d]]$.
- For any band module $B_d(\lambda)$ where $\lambda^2 \neq -1$, $dim_k Ext^1_\Lambda(B_d(\lambda), B_d(\lambda)) = d$ and $R(\Lambda, B_d(\lambda)) \cong k[[t_1, \ldots, t_d]]$.

Current Projects

Current and future problems:

- Finish the case where we have a band module $B_l(\lambda)$ where $\lambda^2 = -1$.
- Determining whether the versal deformation rings are also universal in the case that $\underline{End}_\Lambda(V) \ncong k$.
- Look at what happens to algebras of tame representation type which are not domestic.

Thank you!