Maximal Subalgebras of Finite-Dimensional Algebras

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Logistics and Set-Up

Conventions:

1. k is a field.
2. Unless otherwise noted, all algebras are associative, unital, and finite-dimensional over k.
3. We say two subalgebras A, A' of B are **conjugate** if there is a k-algebra automorphism α of B such that $\alpha(A) = A'$.
4. A subalgebra $A \subset B$ is **maximal** if $A \neq B$ and for any subalgebra A', $A \subset A' \subset B$ implies $A = A'$ or $A' = B$.

For a fixed algebra B, we wish to answer:

1. Can we classify the maximal subalgebras of B?
2. Does B share any interesting representation-theoretic information with its maximal subalgebras?

Previous Work

Related and Similar Results

- Upper bounds for maximal commutative subalgebras of $M_n(\mathbb{C})$: Schur [24]; Gerstenhaber [7].
- Upper bound for maximal subalgebras of $M_n(\mathbb{C})$: Agore [1].
- Classification for maximal subalgebras of central separable algebras: Racine [20], [21].
- Classical work in Lie Theory: Dynkin [3]; Malcev [14].
- Maximal subalgebras of other non-associative algebras: Elduque [4], [5]; Martinez and Zelmanov [15]; Racine [22].

To the best of our knowledge...

- No classification of maximal subalgebras in the general semisimple case.
- No attempt to classify maximal subalgebras of arbitrary finite-dimensional associative algebras.
- No attempt to see if “maximality” has representation-theoretic consequences for finite-dimensional associative algebras.

Classification: Split vs. Semisimple

Fix an algebra B. Let A be a maximal subalgebra of B.

Split Type

A is said to be of **split type** if there exists an ideal of I of B contained in $A \cap J(B)$, such that $J(B)/I$ is a simple A-bimodule and B/I is isomorphic to the trivial extension of A/I by $J(B)/I$.

Semisimple/Separable Type

A is said to be of **semisimple (or separable) type** if $J(B) \subset A$.

Classification: Split vs. Semisimple (Cont.)

Theorem (I-S 2017)

Let B be a finite-dimensional k-algebra, $A \subset B$ a maximal subalgebra. Then A is either of split type or separable type.

Note

If $B/J(B)$ is separable over k, and $B_0 \cong B/J(B)$ is a subalgebra of B such that $B = B_0 \oplus J(B)$, then one of the following holds:

1. If A is of split type, then A is conjugate under an inner automorphism to an algebra of the form $B_0 \oplus I$, where $I \subset J(B)$ is a maximal B-subbimodule (and in particular, $J(B)^2 \subset I$).
2. If A is of separable type, then A is conjugate under an inner automorphism to an algebra of the form $A' \oplus J(B)$, where A' is a maximal subalgebra of B_0.

Classification: Split vs. Semisimple (Cont.)

Maximal Sub-bimodules of J(B)

1. Difficult to classify (non-unital) subalgebras of (non-unital) nilpotent algebras.
2. Results in small dimension [8].
3. Nice interpretation when $k = \overline{k}$ and B is basic.

Maximal Subalgebras of Semisimple Algebras

1. We classify maximal subalgebras of semisimple algebras in the general case.
2. Extends previous work of Racine [21] (central separable case).

Classification: Simple Case

1. Let $B = M_n(D)$, D a division k-algebra with $\dim_k D < \infty$.
2. There are three types of maximal subalgebras of B.
3. The first two were noticed by Racine (the third type does not occur in the central separable case).

Subalgebra Types

1. **S1**: Conjugate to block-upper-triangular matrices with two blocks, contain $Z(D)$.
2. **S2:** Simple, contain $Z(D)$.
3. **S3:** Simple, do not contain $Z(D)$.

Theorem (I-S 2017)

Let $B = M_n(D)$ as before, and $A \subset B$ a maximal subalgebra. Then A is conjugate under an inner automorphism to a maximal subalgebra of type S1, S2, or S3.

Classification: Semisimple Case

Quick Note

The image of the diagonal map $M_n(D) \to M_n(D) \times M_n(D)$ is a maximal subalgebra of $M_n(D) \times M_n(D)$, which we call $\Delta^2(n, D)$.

Theorem (I-S 2017)

Let $B = \prod_{i=1}^{t} M_{n_i}(D_i)$, where each D_i is a division k-algebra, and $n_1 \leq n_2 \leq \ldots \leq n_t$. Then any maximal subalgebra of B is conjugate to an algebra of one of the following two types:

1. *There is an $i < t$ such that $n_i = n_{i+1}$ and $D_i \cong D_{i+1}$, and $A = \left(\prod_{j<i} M_{n_j}(D_j)\right) \times \Delta^2(n_i, D_i) \times \left(\prod_{j>i+1} M_j(D_j)\right)$.*
2. *For some $i \leq t$, $A = \left(\prod_{j<i} M_{n_j}(D_j)\right) \times A_i \times \left(\prod_{j>i} M_{n_j}(D_j)\right)$, where $A_i \subset M_{n_i}(D_i)$ is a subalgebra of type S1, S2, or S3.*

If each block is central simple over k, then this automorphism can be chosen to be an inner automorphism.

Some Consequences

Assume for this slide that $k = \overline{k}$ (of course some of these statements hold in far greater generality!).

Observations from Classification

1. If $B \cong kQ/I$ is basic, the classification works very well.
    1. Split Type: Codim-1 subspaces of parallel arrows.
    2. Separable Type: Identifying vertices of Q.
2. Maximal subalgebras of semisimple algebras are representation-finite and have $\operatorname{gl.dim} \leq 1$.
3. If $B/J(B) = \prod_{i=1}^{t} M_{n_i}(k)$ and $n_1 \leq \ldots \leq n_t$, then the maximal dimension of a subalgebra is $\dim_k B - 1 - \max\{n_1 - 2, 0\}$ (extends [1]).
4. Can "find all maximal subalgebras" using inner automorphisms.
5. Can construct separable and split (in fact trivial) extensions.

Separable and Split Extensions

Go back to k arbitrary.

Separable Functors

1. **Definition:** A functor $F : \mathcal{C} \to \mathcal{D}$ is **separable** if $\eta := \left(X \xrightarrow{\alpha} Y\right) \mapsto \left(F(X) \xrightarrow{F(\alpha)} F(Y)\right)$ has a section, i.e. a natural transformation σ with $\sigma \circ \eta = 1_{\mathcal{C}(-,-)}$.
2. $A \subset B$ is a **separable extension** $\Leftrightarrow$ Res^B_A is separable.
3. $A \subset B$ is a **split extension** $\Leftrightarrow$ $- \otimes_A B$ is separable.

General Principle

1. Separable extensions allow you to transfer representation-theoretic properties from A to B.
2. Split extensions allow you to transfer representation-theoretic properties from B to A.

Maximal Subalgebras and Separable Functors

Theorem (I-S 2017)

Let B be a finite-dimensional k-algebra, and $A \subset B$ a maximal subalgebra of separable type. If $B/J(B)$ is a separable k-algebra, then $A \subset B$ is separable.

Theorem (I-S 2017)

Let B be a finite-dimensional k-algebra, and $A \subset B$ a maximal subalgebra of split type. Suppose that $I_1 \subsetneq I_2$ is a minimal extension of ideals with $I_1 \subset A$ and $I_2 \not\subset A$. Then the following hold:

1. *If $(I_2/I_1)^2 = 0$, then $A/I_1 \subset B/I_1$ is trivial.*
2. *If the simple B-modules are 1-dimensional, $A/I_1 \subset B/I_1$ is split.*

Current/Future Work

Question

Does the poset of subalgebras of B say anything about its representation theory? In particular, does "maximality" mean anything?

Some Notation

1. $\mathcal{P}_k(B)$ = poset of k-subalgebras of B.
2. $\mathrm{Iso}_k(A, B) = \{A' \in \mathcal{P}_k(B) \mid A' \cong A\}$.
3. AbCat = small abelian categories with additive functors.

Restriction and Induction

1. Restriction and induction yield functors $\mathrm{Res} : \mathcal{P}_k(B)^{\mathrm{op}} \to \mathrm{AbCat}$, $\mathrm{Ind} : \mathcal{P}_k(B) \to \mathrm{AbCat}$.
2. Fact: There may be $A, A' \in \mathcal{P}_k(B)$ with $A \cong A'$, which yield non-isomorphic functors $\text{Mod-}A = \text{Mod-}A' \leftrightarrow \text{Mod-}B$.

Current/Future Work (Cont.)

Question

If we fix $A \in \mathcal{P}_k(B)$, can we classify the different functors between Mod-A and Mod-B which arise from Res/Ind in this fashion?

Automorphisms and Embeddings

1. $\mathrm{Aut}_k(B)$ acts on $\mathcal{P}_k(B)$ in a natural way, and $\mathrm{Iso}_k(A, B)$ is an $\mathrm{Aut}_k(B)$-invariant subset.
2. If $A \in \mathcal{P}_k(B)$ has the property that $\mathrm{Iso}_k(A, B) = \mathrm{Aut}_k(B) \cdot A$, then there is only one Res/Ind functor (up to isomorphism).
3. More generally, if $\mathrm{Iso}_k(A, B)$ is a finite union of $\mathrm{Aut}_k(B)$-orbits, then only finitely-many functors arise.

Current/Future Work (Cont.)

Geometric Interpretation

1. For $m \leq n := \dim_k B$, the $\mathrm{GL}(B)$-action on $\mathrm{Gr}_m(B)$ restricts to the closed subgroup $\mathrm{Aut}_k(B)$.
2. $\mathrm{AlgGr}_m(B)$ = $\{m$-dimensional subalgebras of $B\}$ form a closed, $\mathrm{Aut}_k(B)$-invariant subset of $\mathrm{Gr}_m(B)$.
3. **Goal:** For any closed subgroup $G \leq \mathrm{Aut}_k(B)$, classify the G-orbits of $\mathrm{AlgGr}_m(B)$ and relate to isoclasses.

Early Results ($k = \overline{k}$, $B = kQ/I$)

1. Finitely-many $\mathrm{Aut}_k(kQ)$-orbits on $\mathrm{AlgGr}_{n-1}(kQ)$.
2. Finitely-many $\mathrm{Inn}_k(B)$-orbits on $\mathrm{AlgGr}_{n-1}(B) \Leftrightarrow Q$ has no parallel arrows.
3. In type $\mathbb{A}$, $\mathrm{Iso}_k(A, kQ) = \mathrm{Aut}_k(kQ) \cdot A$ for all $A \in \mathrm{AlgGr}_{n-1}(kQ)$.

Thanks

Thanks for listening!

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