On Kontsevich Automorphisms and Quiver Grassmannians

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- $P(z) \in \mathbb{k}[z]$ – any polynomial
- $F_P : \mathbb{K} \to \mathbb{K}$ – algebra automorphism defined by

$$F_P : \begin{cases} X \mapsto XYX^{-1} \\ Y \mapsto P(Y)X^{-1} \end{cases}$$

- Let $P_1, P_2 \in \Bbbk[z]$ be monic polynomials with $P_i(0) = 1$, say

$$P_1(z) = p_{1,0} + p_{1,1}z + \cdots + p_{1,d_1-1}z^{d_1-1} + p_{1,d_1}z^{d_1}$$
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- Take $p_{1,i} = 0 = p_{2,j}$ for $i, j < 0$, $i > d_1$, $j > d_2$.
- Set $\mathbb{A}_+ = \mathbb{Z}_{\geq 0}[p_{1,i}, p_{2,j} : 0 < i < d_1, 0 < j < d_2]$ and call this the *pseudo-positive semiring* associated to P_1 and P_2.

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- For $k \in \mathbb{Z}$, define

$$P_k(z) = \begin{cases} z^{d_2}P_2(z^{-1}) & \text{if } k \equiv 0 \mod 4 \\ P_1(z) & \text{if } k \equiv 1 \mod 4 \\ P_2(z) & \text{if } k \equiv 2 \mod 4 \\ z^{d_1}P_1(z^{-1}) & \text{if } k \equiv 3 \mod 4 \end{cases}$$

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For $k \geq 1$, the elements

$$X_k := F_{P_1} F_{P_2} \cdots F_{P_k}(X) \quad \textit{and} \quad Y_k := F_{P_1} F_{P_2} \cdots F_{P_k}(Y)$$

are pseudo-positive non-commutative Laurent polynomials in X and Y, i.e. are contained in $\mathbb{A}_+\langle X^{\pm 1}, Y^{\pm 1}\rangle \subset \mathbb{K}$.

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$$e \neq v \quad \text{and} \quad |he \cap V| = \sum_{h' \in he \cap H} \omega(h')$$

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Theorem

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$$\gamma_\omega(e, e') = \begin{cases} -d_1 d_2 & \text{if } e \in supp(\omega|_H) \text{ and } e' \in supp(\omega|_V) \\ d_1 & \text{if } e \in supp(\omega|_H) \text{ and } e' \in H \setminus supp(\omega|_H) \\ d_2 & \text{if } e \in V \setminus supp(\omega|_V) \text{ and } e' \in supp(\omega|_V) \\ 0 & \text{otherwise} \end{cases}$$

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- Combinatorial description of closures of cells?

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- If so, the cells should be labeled by compatible gradings ω with the dimension of the cell corresponding to ω given by γ_ω.

Theorem (R.-Weist, coming soon)

Each quiver Grassmannian $Gr_{e_1,e_2}(P_k)$ has such a decomposition into affine cells.

- Schubert-like conditions cutting out these cells?
- Combinatorial description of closures of cells?
- Intersection theory?

Thank you!