Trace Modules and Rigidity

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Notation

In what follows:

R is a local commutative Noetherian ring

M, X are finitely generated R-modules

$\mathrm{Hom}_R(M, X)$ denotes the set of R-linear homomorphisms from M to X

A History of Conjectures

Conjecture (Generalized Nakayama Conjecture, 1975)

Let Λ be an Artin Algebra.
Then any indecomposable injective Λ-module appears as a direct summand in the minimal injective resolution of Λ.

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Equivalently:

Conjecture (Auslander-Reiten Conjecture (ARC), 1975)

Let Λ be an Artin Algebra and M a finitely generated Λ-module
If $\operatorname{Ext}^i_\Lambda(M, M) = 0 = \operatorname{Ext}^i_\Lambda(M, \Lambda)$ *for all* $i > 0$
then M is projective

ARC in Commutative Algebra

Conjecture (ARC, 1993)

Let R be a commutative Noetherian ring and M a fin. gen. R-module.

If $\operatorname{Ext}^i_R(M, M) = 0 = \operatorname{Ext}^i_R(M, R)$, *for all* $i > 0$ *then* M *is projective*

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- Complete Intersection rings (Auslander, Reiten, Solberg - 1993)

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Focus: Rigid Modules

Definition

We call M n-rigid if $\operatorname{Ext}_R^i(M, M) = 0$ for all $1 \leqslant i \leqslant n$.

We call M rigid if $\operatorname{Ext}_R^1(M, M) = 0$.

Example: Rigid Module

Let $R = k[[x,y]]/(xy)$ and $M = R/(x)$.

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Apply $\mathrm{Hom}_R(-, R/(x))$:

$$0 \longrightarrow \mathrm{Hom}_R(R, R/(x)) \xrightarrow{x} \mathrm{Hom}_R(R, R/(x)) \xrightarrow{y} \mathrm{Hom}_R(R, R/(x))$$

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Mult by y is injective on $R/(x)$, so

$$\operatorname{Ext}^1_R(R/(x), R/(x)) = 0.$$

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Some Behaviour of Rigid Modules

$$\begin{tikzcd} M \arrow[rr, "\bar{\alpha}\neq 0"] \arrow[dr, dashed, "\alpha"'] & & X/M \\ & X \arrow[ur, dashed, "\pi"'] & \end{tikzcd}$$

Observe: $\exists\ \alpha : M \to X$ such that $\mathrm{Im}(\alpha) \not\subseteq M$

Compare Trace Modules

For R-modules M and X

Definition

The trace (module) of M in X is

$$\tau_M(X) := \sum_{\alpha \in \mathrm{Hom}_R(M,X)} \alpha(M)$$

We call $\tau_M(R)$ the trace ideal of M.

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Focus: proper trace modules $M = \tau_M(X) \subsetneq X$.

Example: Trace Ideals

Field k,

$$R = k[x,y,z]/(y^2 - xz, x^2y - z^2, x^3 - yz) \text{ with } I = (x,y)$$

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Classes of Examples:

(1) All ideals of grade $\geqslant 2$

(2) All ideals when R is Artinian Gorenstein

Unique Behaviour of Trace Modules

$$\tau_M(X) \xrightarrow{\alpha} X$$

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$$\begin{tikzcd} M^n \arrow[r, two heads] \arrow[rr, bend right, "\in \bigoplus_{i=1}^{n} \mathrm{Hom}_R(M,X)"'] & \tau_M(X) \arrow[r, "\alpha"] & X \end{tikzcd}$$

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Observe: $\mathrm{Im}(\alpha) \subseteq \tau_M(X)$ for all $\alpha \in \mathrm{Hom}_R(\tau_M(X), X)$

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i.e. $\mathrm{Hom}_R(\tau_M(X), X) = \mathrm{Hom}_R(\tau_M(X), \tau_M(X))$

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$$M \text{ a trace module} \Longrightarrow M \text{ cannot map outside of itself}$$

Idea: A proper submodule cannot be both rigid and trace

A Result

Lemma

If $M \subsetneq X$ *is a proper trace module such that* $\operatorname{Hom}_R(M, X/M) \neq 0$ *then* $\operatorname{Ext}^1_R(M, M) \neq 0$ *(i.e.* M *is* **not** *rigid)*

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$$0 \longrightarrow M \longrightarrow X \xrightarrow{\pi} X/M \longrightarrow 0$$

Apply $\mathrm{Hom}_R(M, -)$

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If $M \subsetneq X$ is a proper trace module such that $\operatorname{Hom}_R(M, X/M) \neq 0$ then $\operatorname{Ext}^1_R(M, M) \neq 0$ (i.e. M is **not** *rigid)*

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$$\operatorname{Hom}_R(M, M) \longrightarrow \operatorname{Hom}_R(M, X) \longrightarrow \operatorname{Hom}_R(M, X/M) \longrightarrow \operatorname{Ext}^1_R(M, M)$$

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Repercussions

Since

- $\mathrm{Hom}_R(M, X/M) \neq 0$ whenever X/M has finite length
- rigidity passes to syzygies and
- all ideals in an Artinian Gorenstein ring are trace ideals

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Theorem

Let R be an local Artinian Gorenstein ring. If M is a nonzero syzygy of a proper trace module, then $\mathrm{Ext}^1_R(M, M) \neq 0$. In particular, if $M = \Omega^n I$ for some nonzero ideal $I \subsetneq R$ and $n \in \mathbb{Z}$, then M is not rigid.

More Repercussions

Recall

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Corollary

Let R be an local Artinian Gorenstein ring. The Auslander-Reiten conjecture holds for positive and negative syzygies of ideals $I \subseteq R$.

Open Questions

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Which modules appear as pos/neg syzygies of ideals?

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What happens over 1-dim Artinian Gorenstein rings?

Thank you for your attention!