Dimer models on cylinders over Dynkin diagrams

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Conference on Geometric methods in Representation Theory

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Notation

- G: simply connected complex algebraic group of type ADE
- P: parabolic subgroup
- G/P: partial flag variety
- $\mathbb{C}[G/P]$: coordinate ring

Overview

Geiss - Leclerc - Schröer

There exists a cluster algebra structure on $\mathbb{C}[G/P]$ using subcategory of modules over a preprojective algebra

Jensen - King - Su

$\mathbb{C}[Gr(k,n)]$ has a categorification via a category of Cohen-Macaulay modules of a certain ring.

Baur - King - Marsh

Gave a combinatorial description of the JKS categorification via dimer models.

Goal

JKS: BKM:

k-subset I of $\{1, 2, \ldots, n\}$

$\swarrow$ $\searrow$

JKS	BKM
CM module M_I	a vertex of a quiver
$\downarrow$	$\downarrow$
$T_D = \oplus_I M_I$	a quiver or dimer model D
$\downarrow$	$\downarrow$
$End_B(T_D)$	Jacobian algebra A_D

Theorem (BKM)

The Jacobian algebra $A_D \cong End_B T_D$.

Want a combinatorial model for cluster structure of double Bruhat cells on Kac–Moody group G.

Definition

Let Q: $\begin{array}{ccc} 1 & \xrightarrow{a} & 2 \\ {\scriptstyle d}\uparrow & & \downarrow{\scriptstyle b} \\ 3 & \xleftarrow[c]{} & 4 \end{array}$ with potential $P = abcd$

Cyclic derivatives, $\partial_a(P) = bcd$, $\partial_b(P) = cda$, $\partial_c(P) = dab$, $\partial_d(P) = abc$

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Jacobian ideal, $J(P) =$ Ideal generated by $\{\partial_a(P), \partial_b(P), \partial_c(P), \partial_d(P)\}$

Jacobian algebra, $A(Q, P) = \mathbb{C}Q/J(P)$

Jacobian algebra

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Superpotential $S = \sum$ anticlockwise cycles - $\sum$ clockwise cycles

Cluster structure on $\mathbb{C}[\mathrm{G}r(k,n)]$

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$End_B(T_D) \quad \cong \quad$ Jacobian algebra A_D

Want a combinatorial model for cluster structure of double Bruhat cells on Kac–Moody group G.

Quivers from double Bruhat cells

A Kac-Moody group G behaves like a semi-simple Lie group.

Fact

In particular, G is a disjoint union of the double Bruhat cells

$$G^{u,v} = B_+ u B_+ \cap B_- v B_-$$

where $u, v \in W$

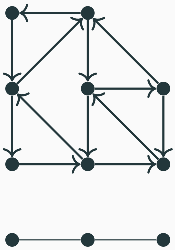
Berenstein, Fomin and Zelevinsky gave a combinatorial way of getting a quiver from double Bruhat cells in Cluster Algebras III.

$$(G, u, v) \rightsquigarrow Q^{u,v} (\textit{call BFZ quiver})$$

Example (BFZ quiver)

Example

$W = S_4$, $u = s_3 s_2 s_1 s_2 s_3$, $v = e$



A_3

Relation to dimers

$$
\begin{array}{ccc}
Gr(k,n) & \longrightarrow & w_n \in S_n \\
\Big\downarrow{\scriptstyle BKM} & & \Big\downarrow{\scriptstyle BFZ} \\
dimer & \xrightarrow{\ \sim\ } & Q^{w_n, e}
\end{array}
$$

In type A, the BFZ quivers are planar, but not true in general.

Quivers in other types

Instead of drawing them on a plane, we will draw the BFZ quivers on the cylinders over the corresponding Dynkin digrams.

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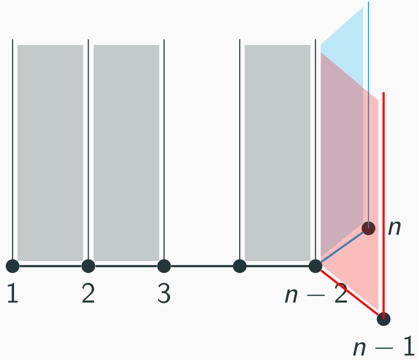
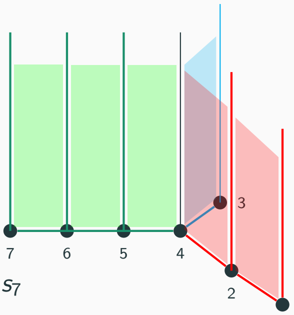
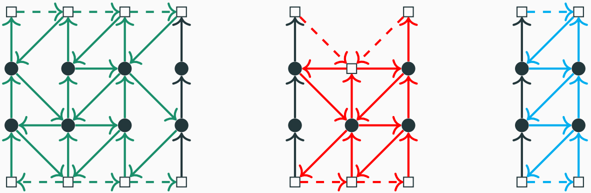


Theorem (K)
For any symmetric Kac–Moody group G, the quiver $Q^{u,v}$ is planar in each sheet.

Example of a dimer model on a cylinder over E_7



$$u = s_1 s_3 s_2 s_4 s_5 s_7 s_3 s_6 s_1 s_5 s_7 s_6 s_4 s_3 s_2 s_1 s_4 s_5 s_6 s_7$$



Quivers in other types

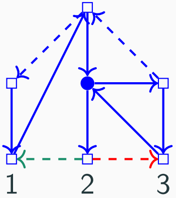
Theorem (K)

- *Each face of $Q^{u,v}$ is oriented.*
- *Each face of $Q^{u,v}$ on the cylinder projects onto an edge of the Dynkin diagram.*
- *Each edge of $Q^{u,v}$ projects onto a vertex of the Dynkin diagram or an edge of the Dynkin diagram.*

The quivers $Q^{u,v}$

To get the quiver $Q^{u,v}$, we attach the quiver $Q^{e,v}$ on top of the quiver $Q^{u,e}$. We will see this with $u = s_1 s_2 s_1 s_3$, $v = s_2 s_3 s_3 s_1 \in S_4$.

$Q^{e,v}$:



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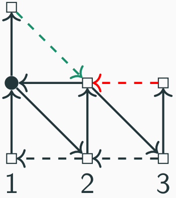

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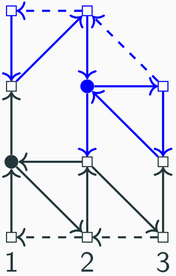

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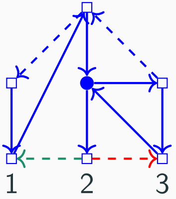

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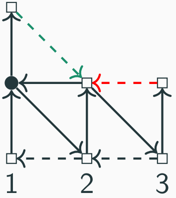

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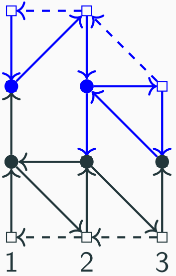

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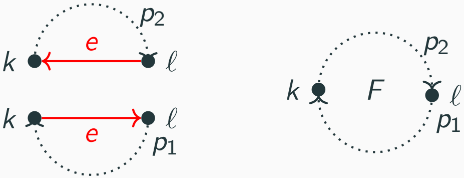

Figure 1: Case 1

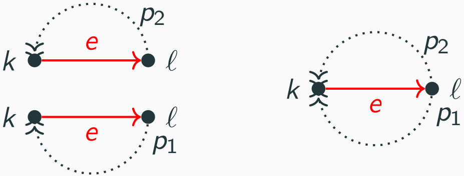

Figure 2: Case 2

Rigid potential

We need the superpotential of these quivers to be Rigid.

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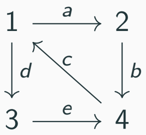
Definition

A potential S is called rigid if every oriented cycle in Q belongs to the Jacobian ideal $J(S)$ up to cyclic equivalence.

Rigid potential

Example (Non-example)



$S_1 = abc$

$J(S_1) = \langle bc, ca, ab \rangle$. So $abc \in J(S_1)$ but $cde \notin J(S_1)$.
Therefore S_1 is not rigid.

Rigid potential

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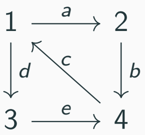

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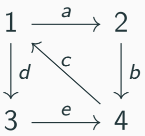
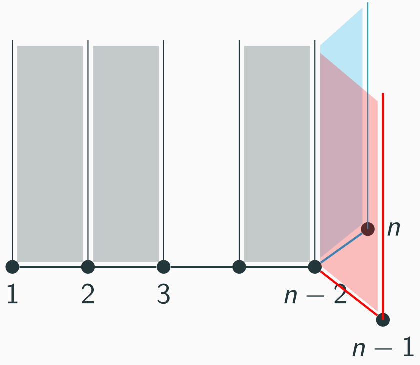
Example



$S_2 = abc + cde$

$J(S_2) = \langle bc, ca, ab + de, ec, cd \rangle$. So $abc \in J(S_2)$ but $cde \in J(S_2)$.
Therefore S_2 is rigid.



Theorem (Buan-Iyama-Reiten-Smith, K)
Let $\mathfrak{g}$ be a simply laced, star shaped Kac-Moody Lie algebra and $Q^{u,e}$ be the quiver corresponding to the double Bruhat decomposition. Then the superpotential of $Q^{u,e}$ is rigid.

Thank you!