Kiyoshi Igusa

ARE FINITE TYPE PICTURE GROUPS VIRTUALLY SPECIAL?

Abstract. This is a report on work of Eric Hanson, a beginning PhD student at Brandeis working under the direction of Corey Bergman and me. Eric is working on combinatorial group theory and representation theory using picture groups and special cube complexes.

For Λ any finite dimensional algebra over any field, one can use τ-tilting theory to define the "picture group" $G(\Lambda)$ of Λ. This group is finitely generated if and only if Λ is τ-tilting finite. In the case Λ is hereditary of finite type, Gordana Todorov and I proved that the picture group is a "CAT(0)"-group by constructing a compact "NPC" cubical space with fundamental group equal to $G(\Lambda)$.

If the cube complex satisfies additional conditions, as pioneered by Haglund and Wise, the space and group are called "special". Picture groups share many of the properties of special groups, so the idea was that they could be special. Eric says this is too optimistic. His conjecture is: "Picture groups of Dynkin quivers are virtually special". Even this weaker condition would imply very nice properties of the picture groups. For example, it would imply that these groups embed in $SL_n(\mathbb{Z})$.

The purpose of this talk is to describe this project and some beginning steps made by Eric Hanson.

1. Introduction

The objective is for Eric to learn current methods in representation theory (τ-tilting theory) and geometric topology (special cube complexes) while working on the following conjecture:

"Picture groups of finite type are virtually special."

- For every artin algebra Λ there is a group $G(\Lambda)$ called the (small) "picture group" of Λ.
- There is also a cubical space $X(\Lambda)$ with
$$\pi_1 X(\Lambda) = G(\Lambda)$$
- **Theorem** [IT1] for Λ hereditary of finite representation type, $X(\Lambda)$ is a $K(\pi, 1)$.
- **Proposition** $G(\Lambda)$ is finitely generated if and only if Λ is τ-tilting finite.
- **Theorem** [HW] (see also [B]) Special groups have very nice properties. For example, they are torsion-free, residually nilpotent, "large" (e.g., they map onto free group on 2 generators) and embed into $SL_n(\mathbb{Z})$.
- (Hanson) $X(KA_2)$ is NOT special. But it has a finite covering which is special. (Recall that finite coverings of a $K(\pi, 1)$ correspond to finite index subgroups of π.) So, $G(KA_2)$ is virtually special.
- In the lecture I pointed out that, if $G(\Lambda)$ is virtually special, it must be torsion-free and embeds in $SL_n(\mathbb{Z})$.

2. Picture group $G(\Lambda)$

Λ a finite dimensional algebra over a field K. There are several definitions of $G(\Lambda)$ using

(1) τ-tilting [DIJ],[BST]
(2) Bridgeland stability conditions ($\hat{G}$) [Br]
(3) Barnard-Carroll-Zhu [BCZ] $\Rightarrow$ Big picture group $\mathbb{P}(\Lambda)$

(4) Treffinger [Tr]⇒ (small) Picture group $G(\Lambda)$

Definition 2.0.1. For Λ an artin algebra, the *big picture group* $\mathbb{P}(\Lambda)$ is the group given by generators and relations as follows.

- Generators are
  1. $g_{\mathcal{T}}$ for all torsion classes $\mathcal{T}$ in *mod*-Λ
  2. x_M for all bricks M ($End_\Lambda(M)$ is a division algebra)
- Relations:
  1. $g_0 = e$
  2. If $\mathcal{T} \subset \mathcal{T}'$ is a minimal inclusion (denoted $\mathcal{T} <\cdot\, \mathcal{T}'$) then
  $$g_{\mathcal{T}} x_M = g_{\mathcal{T}'}$$
  where M is the minimal object of $\mathcal{T}'$ not in $\mathcal{T}$. (BCZ: M is unique, a brick and all bricks occur in this way.)

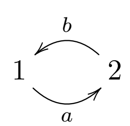
Definition 2.0.2. The (small) picture group $G(\Lambda)$ has:

- Generators are
  1. $g_{\mathcal{T}}$ for functorially finite torsion classes $\mathcal{T}$
  2. x_M for all functorially finite bricks (f-bricks) M (i.e., so that $Gen(M)$ is functorially finite.)
- Relations:
  1. $g_0 = e$
  2. If $\mathcal{T} <\cdot\, \mathcal{T}'$ is a minimal inclusion of f.f. torsion classes then
  $$g_{\mathcal{T}} x_M = g_{\mathcal{T}'}$$
  where M is the minimal object of $\mathcal{T}'$ not in $\mathcal{T}$. (Treffinger: M is an f-brick and all f-bricks occur in this way. And I understand that [BCZ] are also preparing a proof of this statement.)

Λ has finite representation type $\Rightarrow \mathbb{P}(\Lambda) = G(\Lambda)$. More generally we have:

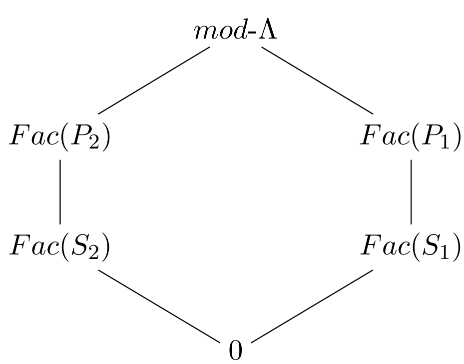
Theorem 2.0.3 (Demonet-Iyama-Jasso, [DIJ])**.** *If Λ is τ-tilting finite (i.e. $G(\Lambda)$ is finitely generated) then all torsion classes are functorially finite. Equivalently, $\mathbb{P}(\Lambda) = G(\Lambda)$.*

Example 2.0.4. Let Λ

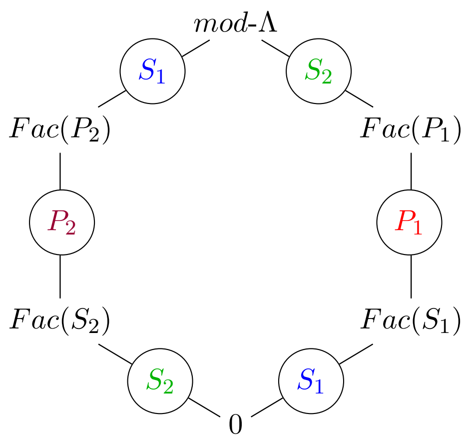


$$ab = 0 = ba$$

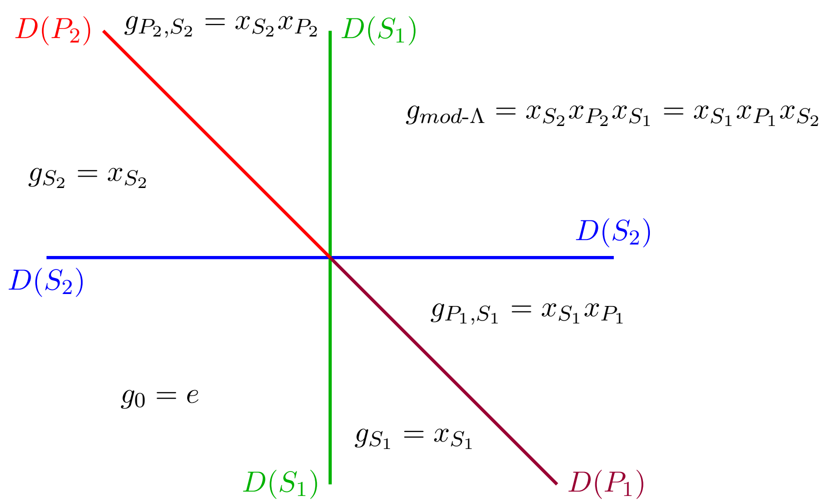
(preprojective algebra of type A_2) There are four modules, all bricks: S_1, S_2, P_1, P_2 and six torsion classes ordered by inclusion:



where $Fac(M)$ is all quotients of objects in $add\, M$.



The edges in the Hasse diagram are minimal inclusions of torsion classes. So, they are labeled with bricks. (Following Iyama, labels are encircled.)

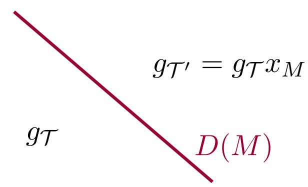


This figure is in $(\mathbb{R}^3)^* = Hom(K_0(\Lambda), \mathbb{R})$

$$D(M) = \{\theta \in (\mathbb{R}^3)^* \mid \theta(M) = 0, \theta(M') \leq 0 \,\forall M' \subset M\}$$

$$\theta = y - x = (-1, 1) \in D(P_2) \text{ since } \theta(P_2) = 0,\ \theta(S_1) = -1 < 0$$

In answer to a question of Ryan Kinser: The g-vector of a τ-rigid module X with projective presentation $P \to Q \to X$ is the point $\theta_X \in Hom(K_0(\Lambda), \mathbb{Z})$ given by $\theta_X(M) = \dim Hom(Q, M) - \dim Hom(P, M)$. For example, $\theta_{S_2} = y - x = (-1, 1) \in D(P_2)$. The regions are simplicial cones labeled $g_{\mathcal{T}}$ where $\mathcal{T} = Fac(T_1 \oplus \cdots \oplus T_k)$ with the g-vectors of the T_i, components of a support τ-tilting object and $n-k$ negative projectives $P_j[1]$ being the vertices of the simplicial cones. Adjacent regions form a minimal inclusion of f.f.torsion classes and the wall separating adjacent regions is $D(M)$ where M is the f-brick label of the minimal inclusion. (See [BST].) Thus the local picture is:



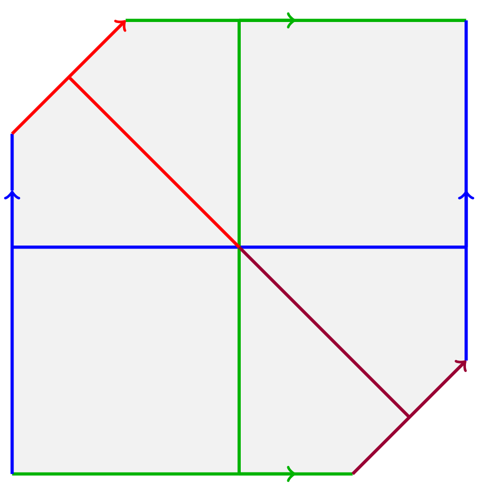
When a letter occurs only once in any relation, that letter and that relation can be eliminated. So, we see that the picture group is the free group on the three letters. $x_{S_1}, x_{S_2}, x_{P_1}$ (or $x_{S_1}, x_{S_2}, x_{P_2}$):

$$G(\Lambda) = \langle x_{S_1}, x_{S_2}, x_{P_1}, x_{P_2} | x_{S_2} x_{P_2} x_{S_1} = x_{S_1} x_{P_1} x_{S_2} \rangle = F_3$$

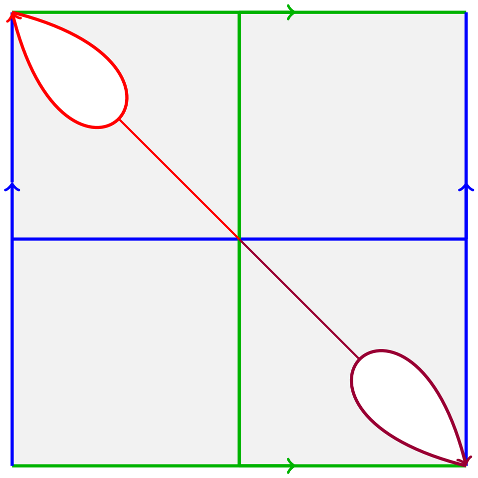
More generally, for Λ τ-tilting finite, $G(\Lambda)$ is generated by x_M for all bricks M modulo the relation that all maximal green sequences are equal where a *maximal green sequence* is the sequence of brick labels of any maximal (finite) chain in the poset of torsion classes.

3. Picture space $X(\Lambda)$

There is an associated topological space which is "cubical" which means it is a union of cubes $[0,1]^n$ where $n = |Q_0|$. (A technical condition which I skipped: the intersection of two cubes is a cube. To meet this condition each of these cubes must be cut into 2^n subcubes [I].) I will illustrate this with the above example. First, make a solid hexagon which is a union of six "squares". Vertices are labeled with torsion classes.



Then, paste together all sides with the same label (color) and paste together all vertices. This is easier to visualize if we first do a couple of vertex identifications:



Then paste together opposite side in the way we all know and get a torus with two boundary components which meet at a point (not able to draw this, so I didn't use slides).

This is well-known to be a $K(\pi, 1)$ with $\pi = F_3 = G(\Lambda)$. This solves the following problem in this case (given the lemma).

Problem 3.0.5 (for Eric)**.** Show that the cubical complex $X(\Lambda)$ is a $K(\pi, 1)$ for the picture group of Λ for any Λ of finite τ-tilting type.

Lemma 3.0.6 (Gromov [G])**.** *A cubical complex is a $K(\pi, 1)$ if and only if the link of every vertex is flag.*

This condition, which used to be called "locally $CAT(0)$", is now known as NPC (*nonpositively curved*).

4. Special cube complexes

The main problem that Eric Hanson is trying to solve is the following.

Problem 4.0.7. Show that the picture space $X(\Lambda)$ is virtually a special NPC cube complex, i.e., it has a finite covering which is special.

This would imply that picture groups of finite type are virtually special. However, it might happen that picture groups are all special. For example, free groups are special. So, even though $X(\Lambda)$ for Λ in the example above, or for Λ the hereditary algebra of type A_2 which is very similar, $X(\Lambda)$ is not special.

Definition 4.0.8 (Haglund-Wise [HW])**.** A cube complex X is *special* if its hyperplanes $H \subset X$ satisfy the following conditions.

1. H does not cross itself
2. H does not "directly self-osculate"
3. No two hyperplanes "inter-osculate" (intersect at one point and osculate at another).

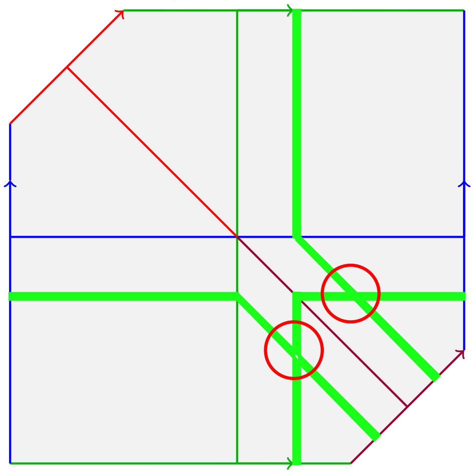
A rigorour way to define H is in terms of its transversal which is the equivalence class of edges (called a "wall") where the equivalence relation is generated by the relation that opposite sides of each square are equivalent.

1. The sides of each square form two equivalence classes. (The corresponding hyperplanes intersect.)

(3) Given two inequivalent sides e, e' of any square, there does not exist a vertex v which lies on edges f, f' equivalent to e, e' so that f, f' do not lie on the same square. (The corresponding hyperplanes osculate at vertex v.)

(2) There does not exist a vertex v which lies on equivalent edges e, e' which do not lie in the same square so that e, e' are both oriented towards v with some consistent orientation of all edges in the equivalence class.

Our example is NOT special. All three conditions fail! The easiest one to see is Conditions (1). (The hyperplane is not in the center of the square: it is 1/4 of the way from one side since each square is actually cut into 4 squares.)



In green is one hyperplane which self-intersects twice!

Theorem 4.0.9 (Hanson)**.** *This cube complex has a finite covering which is special.*

A finite covering corresponds to a finite index subgroup of $\pi_1 X(\Lambda) = G(\Lambda)$. Since Hanson's proof is nonconstructive, we don't know which subgroup it is. However, in this case the entire group $G(\Lambda) = F_3$ is special (all free groups are special). But the cube complex $X(\Lambda)$ is not special. My belief is that $G(\Lambda)$ is always special. There should be another "good" cube complex $X'(\Lambda)$ which is special and which is homotopy equivalent to $X(\Lambda)$.

References

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