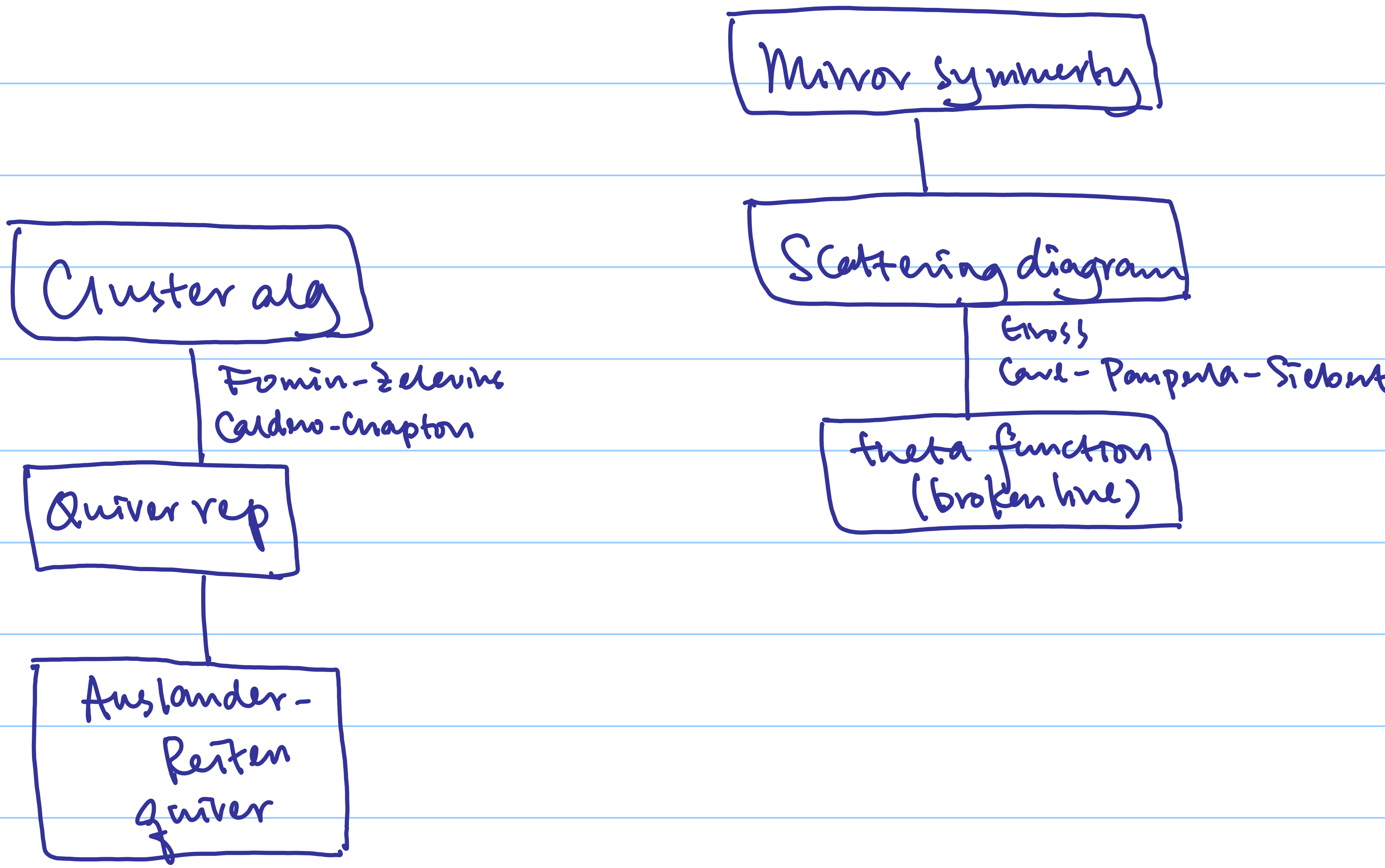
Quiver Representations and Theta Functions

Man Wai Cheung

$$\begin{array}{c}\boxed{\text{Cluster alg}}\\ \Big| \quad \text{Fomin-Zelevinsky} \\ \Big| \quad \text{Caldero-Chapoton} \\ \boxed{\text{Quiver rep}} \\ \Big| \\ \boxed{\text{Auslander-Reiten quiver}}\end{array}$$


Mirror symmetry
Scattering diagram
Gross
Carl-Pumperla-Siebert
theta function
(broken line)
Cluster alg
Fomin-Zelevinsky
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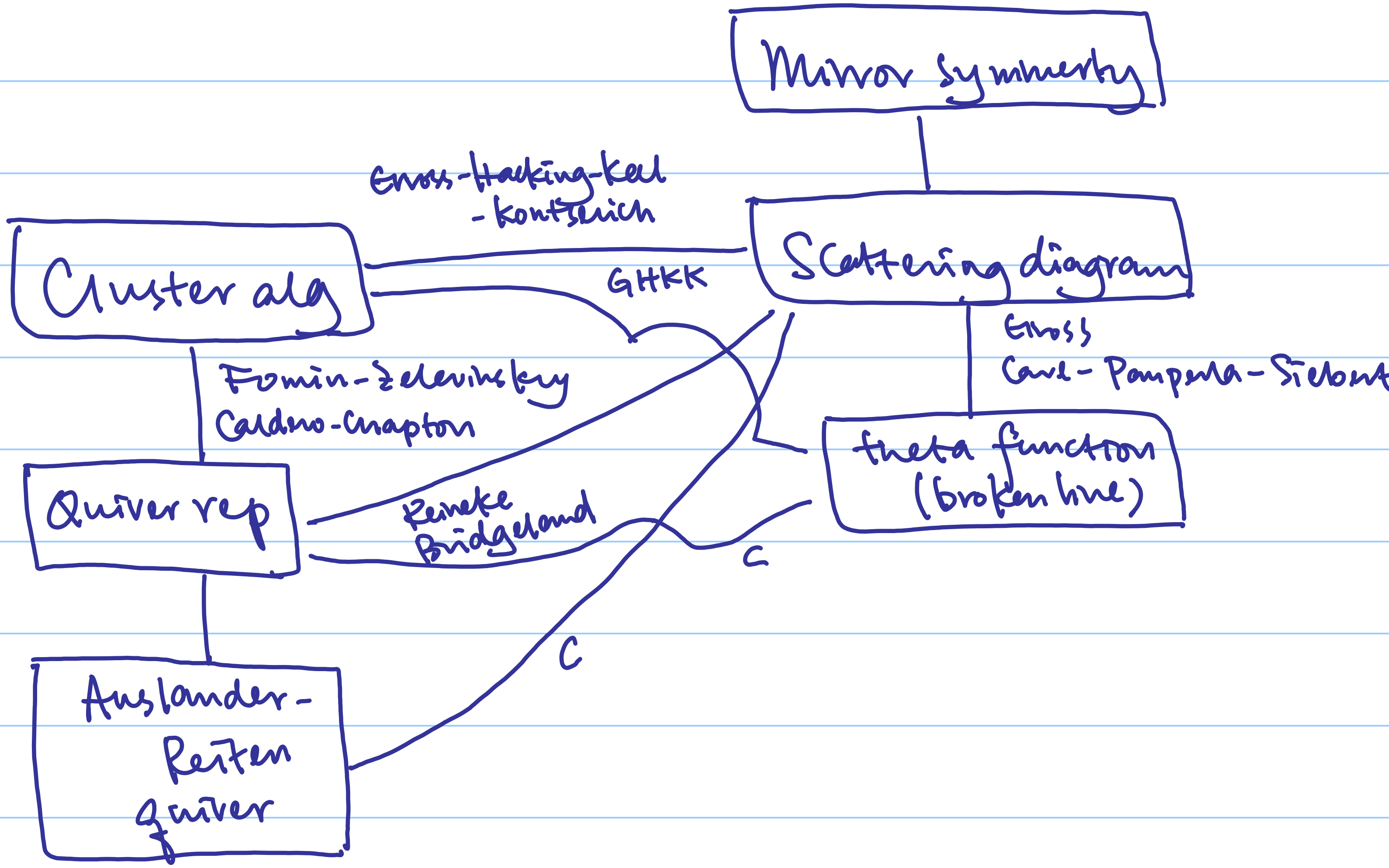
Quiver rep
Auslander-Reiten quiver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-Theta functions: [Gross-Hacking-Keel-Siebert] use to construct the mirror to an arbitrary log Calabi-Yau surface.
~describe counting of tropical curves

Quiver representation

Q : (acyclic) quiver of rank n with: Q_0: set of vertices, Q_1: set of arrows

dim vector of quiver rep.

Let $N = \mathbb{Z}Q_0$ and $M = \mathrm{Hom}(N, \mathbb{Z})$, i.e. $n = \#$ vertices in Q.

Now, we define $\{\cdot,\cdot\}$ on N as

$$\{e_i, e_j\} = \{\#\text{arrow from } i \to j\} - \{\#\text{arrow from } j \text{ to } i\}$$

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$$\{e_i,e_j\} = \{\#\text{arrow from } i\to j\} - \{\#\text{arrow from } j \text{ to } i\}$$

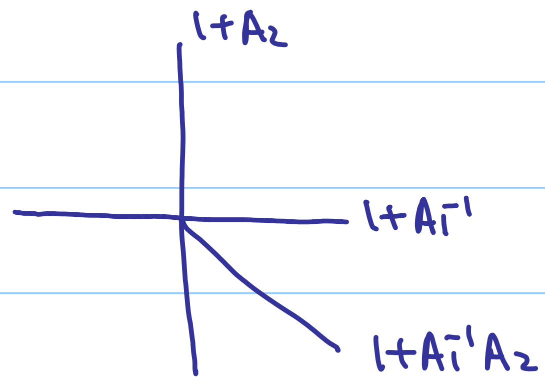
Define a bilinear form $\chi(\cdot,\cdot)$, the Euler form, on N as

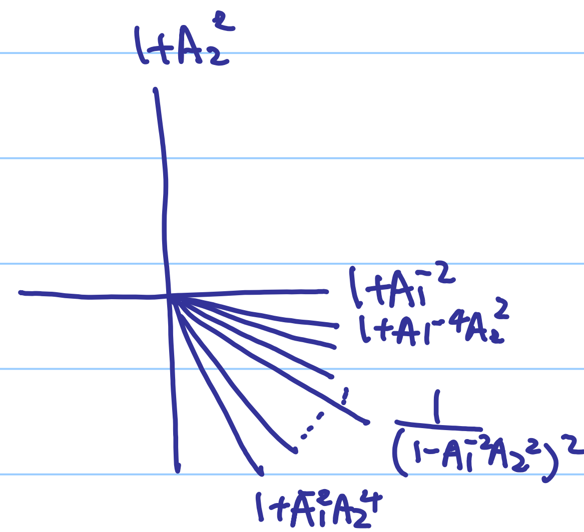
$$\chi(d,e) = \sum_{i\in Q_0} d_i e_i - \sum_{\alpha: i\to j} d_i e_j \qquad d,e\in N$$

Define $\varepsilon: N\to M$ by $\qquad \varepsilon(d) = \chi(\cdot, d)$

Scattering diagram

E.g.





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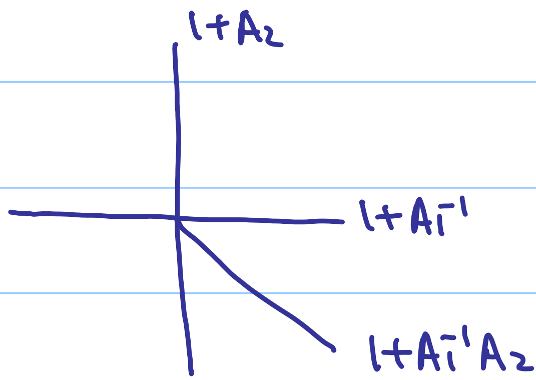

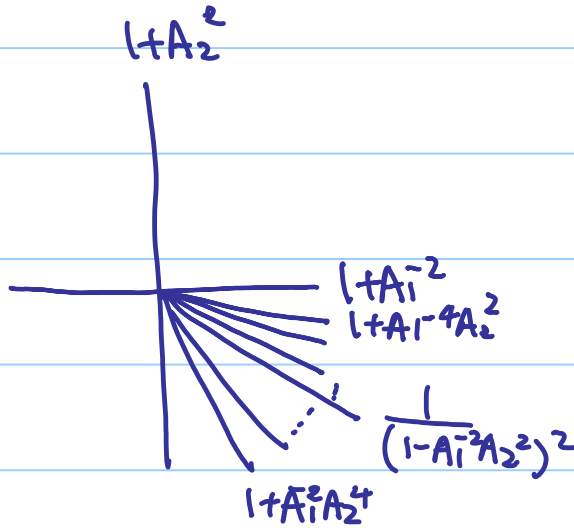

<u>Note A_i is nothing but x_i</u>

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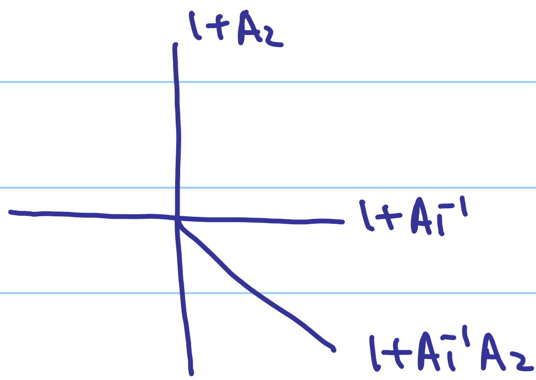

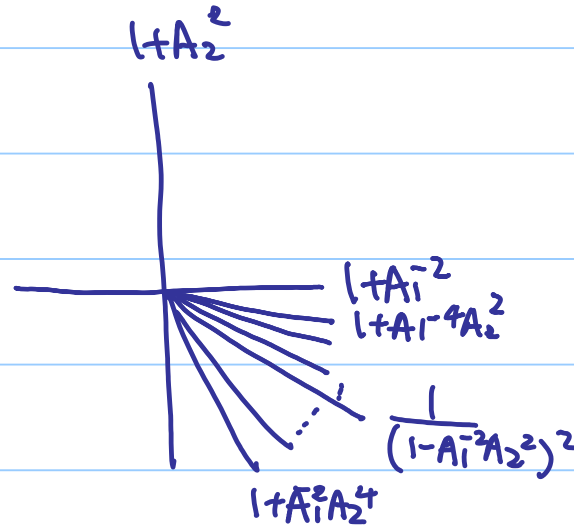

<u>Note A_i is nothing but x_i</u>

for $m=(m_1,\dots,m_n)$ write $A^m = A_1^{m_1}\cdots A_n^{m_n}$

Define $p^*: N \to M$

$n \mapsto \{n, \cdot\}$

Wall (d, f_d)

- $d \subseteq M_{\mathbb{R}}$, support of the wall, is a convex rational polyhedral cone of codimension one, $d \subseteq n^{\perp}$ for some $n \in N^{+}$ (i.e. $n_i \geq 0 \ \forall i$)
- $f_d \in \mathbb{C}[A_1, \dots A_n]$ s.t. $f_d = 1 + \sum_{k \geq 1} c_k A^{k p^*(n)}$ for some $c_k \in \mathbb{C}$

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__Def__ A scattering diag. $\mathfrak{D}$ is a collection of walls such that for each $k \geq 0$, the set

$$\{(d, f_d) \mid f_d \not\equiv 1 \mod (A_1 \dots A_n)^k\} \text{ is finite.}$$

Consistent condition

A wall (d, f_d), $d \subseteq n^\perp$

i.e. $\langle n, w \rangle$ for all $w \in d$

"think w as a stability condition"

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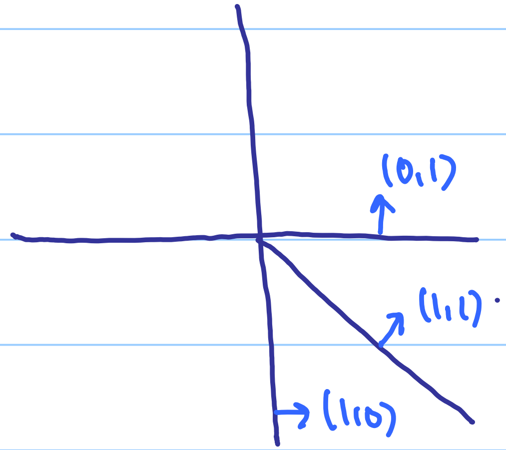
<u>Thm</u> (Bridgeland) Hall alg. scattering diag.

- Walls consists of maps $w \in M_{\mathbb{R}}$ s.t.

  $\exists$ w-semistable obj. in rep(Q)

$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet}$

E.g.



$\rightsquigarrow$

$0 \to \mathbb{C}$

$\mathbb{C} \to \mathbb{C}$

$\mathbb{C} \to 0$

$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet}$

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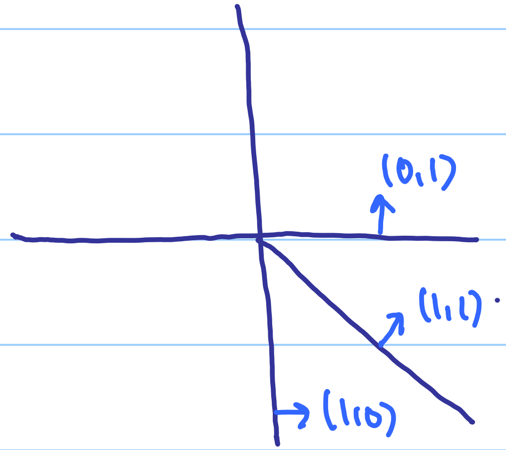

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Auslander-Reiten theory

A way to "line up" irreducible rep. of a quiver.

e.g. $\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \rightsquigarrow$

$$\begin{tikzcd} & \mathbb{C} \to \mathbb{C} \arrow[dr] & \\ 0 \to \mathbb{C} \arrow[ur] & & 0 \to \mathbb{C} \end{tikzcd}$$

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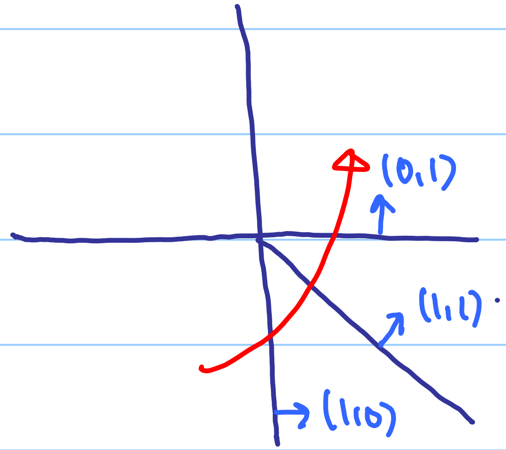

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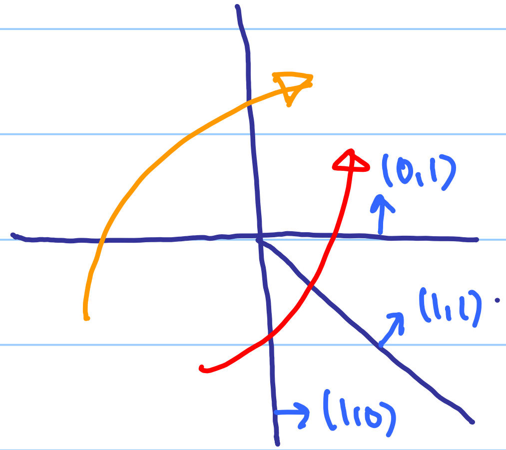

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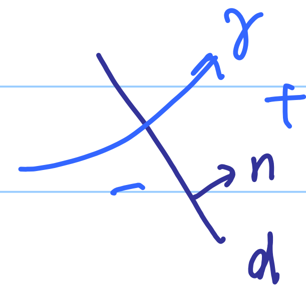
\end{tikzcd}$$

γ : path in $\mathcal{D}$

Good crossing



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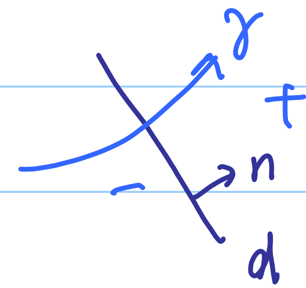

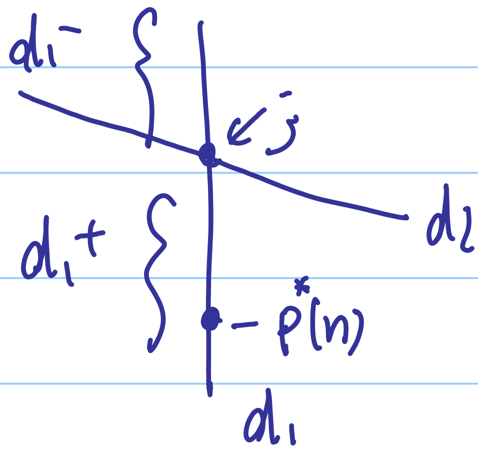

$d_1 = d_1^+ \cup j \cup d_1^-$

Let D_1, D_2 be the indecomposable rep. associated to d_1, d_2

<u>Thm</u>('16) If γ goes from d_1^+ to d_2, and the two crossing are good, then D_2 is a predecessor of D_1

"reverse the order in Auslander-Reiten quiver"

Theta function - given by broken line

Fix. $\mathfrak{D}$: scattering diag.

$m \in M \setminus \{0\}$

$Q \in \{ m \in M_{\mathbb{R}} \mid \langle m, e_i \rangle > 0 \ \forall i \}$ positive chamber

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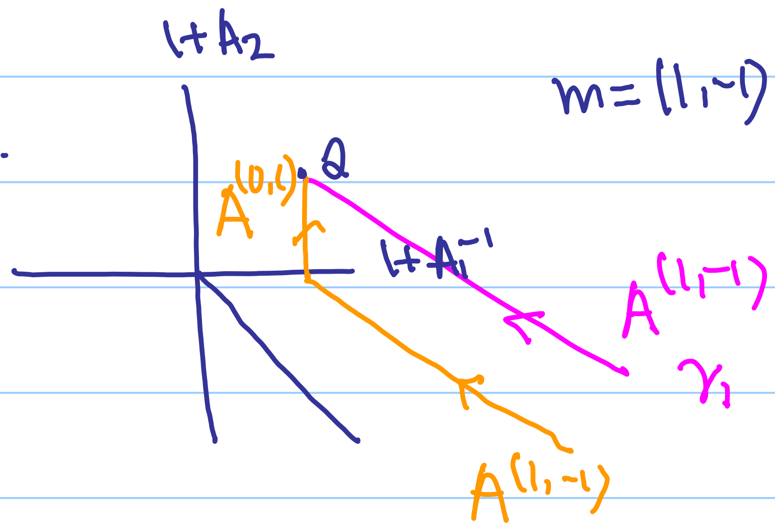

$m = (1,-1)$

Two broken lines: γ_1, γ_2

$$\vartheta_{m,Q} = A^{(1,-1)} + A^{(-1,-1)}$$

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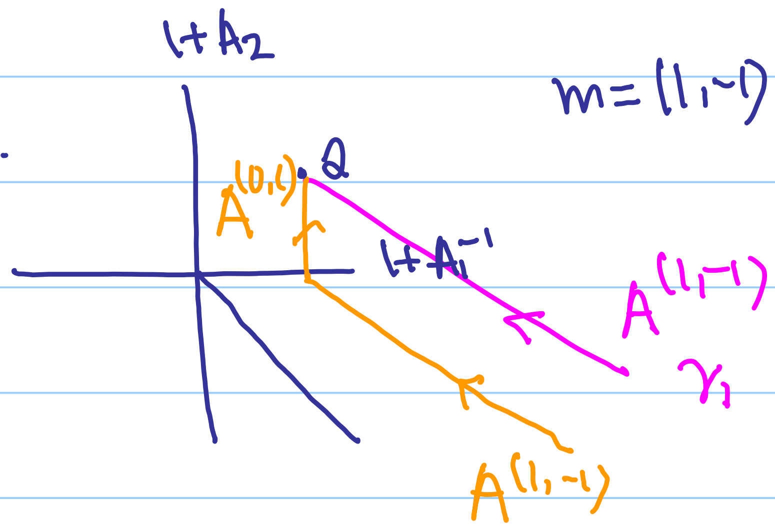

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Thm (GHKK) If $\vartheta_{m,Q}$ is a finite sum, then $\vartheta_{m,Q}$ is an elt. of cluster alg.

Thm (Caldero-Chapoton) Let Q be a finite quiver with vertices $1, \dots, n$. and D a f.d. repn of Q with dimension vector d. For $e \in \mathbb{N}$,

denote $Gr(e, D) := \{ E \in \mathrm{mod}(Q) \mid E \subseteq D, \dim(E) = e \}$

Define

$$CC(D) = \frac{1}{A_1^{d_1} \cdots A_n^{d_n}} \sum_{0 \leq e \leq d} \chi(Gr(e, D)) \prod_{i=1}^{n} A_i^{\sum_{j \to i} e_j + \sum_{i \to j} (d_j - e_j)}$$

Then $CC(D) = X_D$ cluster variable obtained from D.

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$\Rightarrow$ this is simply theta function

$$CC(D) = \frac{1}{A^d} \sum_{0 \le \ell \le d} \chi\left(Gr(\ell, D)\right) \prod_{i=1}^{n} A_i^{\sum_{j \to i} \ell_j + \sum_{i \to j} (d_j - \ell_j)}$$

$$= \frac{\prod A_i^{\sum_{i \to j} d_j}}{\prod_i A_i^{d_i}} \sum_{0 \le \ell \le d} \chi\left(Gr(\ell, D)\right) \prod_{i=1}^{n} A_i^{\sum_{j \to i} \ell_j + \sum_{i \to j} -\ell_j}$$

$$= A^{-\varepsilon(d)} \sum_{0 \le \ell \le d} \chi\left(Gr(\ell, D)\right) \prod_{i=1}^{n} A_i^{p^*(\ell)}$$

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initial slope

$-\mathcal{E}(d) + p^*(\ell)$

final slope

$$CC(D) = \frac{1}{A^d} \sum_{0 \le e \le d} \chi(Gr(e,D)) \prod_{i=1}^{n} A_i^{\sum_{j\to i} e_j + \sum_{i \to j} (d_j - e_j)}$$

$$= \frac{\prod A_i^{-\sum_{i\to j} d_j}}{\prod_i A_i^{d_i}} \sum_{0 \le e \le d} \chi(Gr(e,D)) \prod_{i=1}^{n} A_i^{\sum_{j \to i} e_j + \sum_{i\to j} -e_j}$$

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$-\mathcal{E}(d) + p^*(e)$

final slope

Understand $\mathcal{E}(d)$ as injective resolution.

$$0 \to D \to I_1 \to I_2 \to 0$$

Look for Hall alg. theta function

alg for the quiver representations

Look for Hall alg. theta function

$\uparrow$ alg for the quiver representations

Theta fcn : $\quad \vartheta_m \triangleq 1_{\mathcal{F}(m)}^{-1} \, A^m \, 1_{\mathcal{F}(m)}$

going to positive chamber

(note opp. to Bridgeland notation)

$T(w) = \{ E \in \text{rep}\, Q :$ any quotient obj. $E \twoheadrightarrow F$ satisfies $w(F) > 0 \}$

$F(w) = \{ E \in \text{rep}\, Q :$ any subobj. A satisfies $w(A) \leq 0 \}$.

Thm (1b) $\mathcal{V}_m = \mathcal{G}_{\mathcal{F}(m)}(D) A^m$, $m = \mathcal{E}(\dim D)$, m in cluster complex

where objects in $\mathcal{G}_{\mathcal{F}(m)}(D)$ are rep'ns $E \in \mathcal{F}(m)$
& E is equipped with an inclusion into D.

Thm (1b) $\quad x_m = \mathcal{G}_{\mathcal{F}(m)}(D) A^m$, $\quad m = \varepsilon(\dim D)$, m in cluster complex

where objects in $\mathcal{G}_{\mathcal{F}(m)}(D)$ are repns $E \in \mathcal{F}(m)$

& E is equipped with an inclusion into D.

apply Euler char χ

$$x_m = A^{-\varepsilon(D)} \sum_{0 \le e \le d} \chi(\mathrm{Gr}(e, D)) A^{p^*(e)}$$

$\Rightarrow$ the CC-formula

$$\vartheta_m = A^{-\varepsilon(D)} \sum_{0 \le \ell \le d} \chi(Gr(\ell, D)) A^{p^*(\ell)}$$ for acyclic quiver

$$\mathcal{V}_m = A^{-\mathcal{E}(D)} \sum_{0 \le \ell \le d} \chi(Gr(\ell, D)) A^{p^*(\ell)}$$ for acyclic quiver

We can generalize this result to quiver (Q, W). $W = \sum$ cycles.

$$\mathcal{U}_m = A^{-\varepsilon(D)} \sum_{0 \le \ell \le d} \chi(Gr(\ell, D)) A^{p^*(\ell)}$$ for acyclic quiver

We can generalize this result to quiver (Q, W). $W = \sum$ cycles.

Now given D, consider $\hat{Q}$ where $\hat{Q} = Q \setminus \{\ell_i\}$

for i is a sink in D,

ℓ_i is the edge pointing to i.

Then we consider the inj. resol. of $D | \tilde{Q}$.

get the same result.

Detour.:

How about Q is not in positive chamber?

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if there exists a path going from m to Q by good crossing.

then $\quad \vartheta_{m,Q}$ is obtained by conj. of $\mathbb{1}_{F(Q)}^{-1}\mathbb{1}_F(m)$ on A^m

Detour.:

How about Q is not in positive chamber?

if there exists a path going from m to Q by good crossing.

then $\quad \mathcal{V}_{m,Q}$ is obtained by conj. of $1_{\mathcal{F}(Q)}^{-1} 1_{\mathcal{F}}(m)$ on A^m

<u>Thm</u> $\quad \mathcal{V}_{m,Q} = \mathcal{G}_{\mathcal{F}(m)\setminus\mathcal{F}(Q)}(D)\, A^m \qquad m = \underline{\mathcal{E}}(D)$

where $\mathcal{G}_{\mathcal{F}(m)\setminus\mathcal{F}(D)}$ contains obj. as rep E s.t. $E \in \mathcal{F}(m)\setminus\mathcal{F}(Q)$

and $\ker(E \to D) \in \mathcal{F}(Q)$

Detour.:

How about Q is not in positive chamber?

if there exists a path going from m to Q by good crossing.

then $\mathcal{V}_{m,Q}$ is obtained by conj. of $l_{F(Q)}^{-1} l_F(m)$ on A^m

Thm $\mathcal{V}_{m,Q} = \mathcal{G}_{F(m)\setminus F(Q)}(D)\, A^m \qquad m = \mathcal{E}(D)$

where $\mathcal{G}_{F(m)\setminus F(D)}$ contains obj. as rep E s.t. $E \in F(m)\setminus F(Q)$

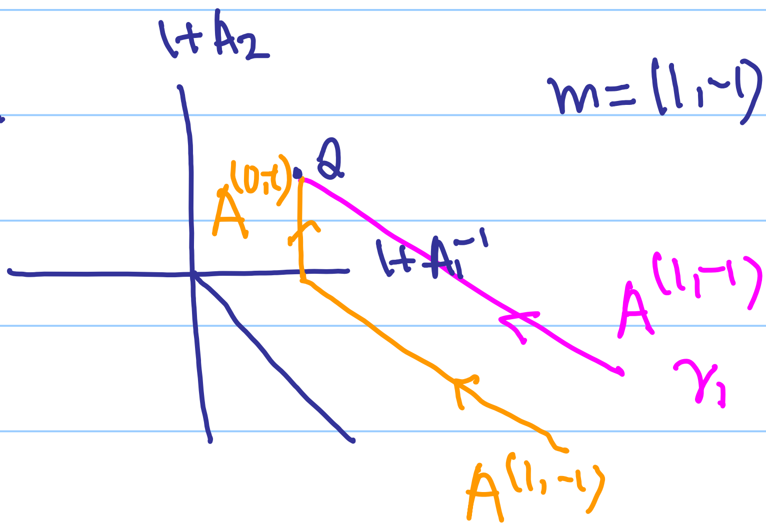
and $\ker(E \to D) \in F(Q)$

If m to Q by negative crossing, replace $\mathcal{G}$ with Extension

switch $\tau \leftrightarrow \omega$

What about broken line?

Ex.



$m=(1,-1)$.

$D = \mathcal{O} \to \mathbb{C}$

$E = \mathcal{O} \to \mathcal{O}$

$\mathcal{O} \to \mathbb{C}$

$0 \subseteq (\mathcal{O} \to \mathbb{C})$

First bending

<u>Thm</u>(-) $pr_{\cdot f_1^\perp}(A^{-\varepsilon(d)}) = \mathcal{G}_1 A^{-\varepsilon(d)}$

where obj. in $\mathcal{G}_1$ are $[F_1^\lambda \to D]$

with no kernel, of dim vector f_1

Apply χ $\rightsquigarrow$ get $\sum_\lambda Gr(\lambda, Hom(F_1 D))$

which agrees with usual wall-crossing

Since mult. λ_1, define $V_1 := F_1^{\lambda_1}$

k-th bending

from (k-1), have $0 \subseteq V_1 \subseteq \cdots \subseteq V_{k-1}$, $V_i / V_{i-1} = F_i^{\lambda_i}$

(-) The k-th bending gives $\mathcal{G}_k$

where Poincare polynomial of $\mathcal{G}_k$ is

$$Gr(\lambda_k \, Hom(F_k, D/V_{k-1}) - Ext(V_{k-1}, F_k)) \times \mathbb{A}^{\lambda_k Ext(V_{k-1}, F_k)}$$

& $0 \subseteq V_1 \subseteq \cdots \subseteq V_{k-1} \subseteq V_k$

$\dim V_k = V_{k-1} + \lambda_k \dim F_k$

THANK YOU!