Virtual Projectivity

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Notation

For the purposes of this talk, please assume that:

- k = a field of characteristic p, algebraically closed,
- G = a finite group,
- all kG-modules are finitely generated.

If M is a kG-module, let $\Omega(M)$ denote the kernel of a projective cover $P_M \to M$ and let $\Omega^{-1}(M)$ be the cokernel of an injective hull $M \to I_M$.

For $n > 1$, let $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ and $\Omega^{-n}(M) = \Omega^{-1}(\Omega^{1-n}(M))$.

Recall that projective modules are injective and *vice versa*.

Things We Need To Know.

For M a kG-module, let $M^* = \mathrm{Hom}_k(M, k)$, the k-dual.

- On the category of kG-modules there is a tensor product $\otimes = \otimes_k$ with action of $g \in G$ on $M \otimes N$ given by $g(m \otimes n) = gm \otimes gn$.
- $\mathrm{Hom}_k(M, N) \cong M^* \otimes N$. $[(\lambda \otimes n)(m) = \lambda(m)n]$
- There is a trace map $\mathrm{Tr} : M^* \otimes M \to k$ given by $\lambda \otimes m \mapsto \lambda(m)$.
- There is a unit map $u : k \to M^* \otimes M \cong \mathrm{Hom}_k(M, M)$ that sends $1 \in k$ to $\mathrm{Id}_M \in \mathrm{Hom}_{kG}(M, M)$

Note that if p does not divide the dimension of M, then Tr is split by the unit map and k is a direct summand of $M^* \otimes M$.

$$k \xrightarrow{\ u\ } M^* \otimes M \xrightarrow{\ \mathrm{Tr}\ } k$$

Theorem: (Benson-Carlson, 1986) Assume that k is algebraically closed. Suppose that M and N are indecomposable modules and that k is a direct summand of $M \otimes N$. Then

1. $\mathrm{Dim}(M)$ is not divisible by p,
2. $N \cong M^*$,
3. the multiplicity of k as a direct summand of $M \otimes N$ is one, and
4. the trace map $\mathrm{Tr} : M \otimes M^* \to k$ is split.

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Corollary Suppose that M and N are kG-modules such that M is indecomposable and has dimension divisible by p. Then any direct summand of $M \otimes N$ has dimension divisible by p.

The Stable Category.

The stable category **stmod**(kG) has

objects: Finitely generated kG-modules

and morphisms (for M and N objects):

$$\underline{\mathrm{Hom}}_{kG}(M, N) = \frac{\mathrm{Hom}_{kG}(M, N)}{\mathrm{PHom}_{kG}(M, N)}$$

where PHom means homomorphisms that factor through projectives modules.

This is a *tensor* triangulated category. The triangles correspond to exact sequences. The shift functor is Ω^{-1}.

Note that $\mathrm{Ext}^n(M, N) \cong \underline{\mathrm{Hom}}_{kG}(\Omega^n(M), N)$.

Definition Suppose that M is a kG-module. We say that a kG-module X is relatively M-projective (or just M-projective) if X is a direct summand of $M \otimes U$ for some kG-module U. We say that a map $\varphi : X \to Y$ is M-projective if it factors through an M-projective module.

Definition: (C-Peng-Wheeler) We say that a module X is virtually M-projective if, for n sufficiently large, any homomorphism $\Omega^n(X) \to X$ factors through an M-projective module. The degree of virtual M-projectivity of X is the least such n.

Lemma: Suppose that $p > 2$, there exists a kG-module M such that k is virtually M-projective, but not M-projective.

Proof: Let n be the least common multiple of the degrees of the nonnilpotent generators $\mathrm{H}^*(G,k) \cong \mathrm{Ext}^*_{kG}(k,k)$. Let

$$M = \Omega^n(k)/S$$

where S is a one-dimensional submodule of $\Omega^n(k)$. Then we check

1. M is indecomposable, (takes a little proof)
2. p divides the dimension of M, (because n is even)
3. every homomorphism $\Omega^n(k) \to k$ factors through M. (because S is in the radical of $\Omega^n(M)$)

So, (1) and (2) imply that k is not M-projective, while (3) implies that it is virtually M-projective.

For M a kG-module, the ring $\mathrm{Ext}^*_{kG}(M, M)$ is a finitely generated module over the cohomology ring $\mathrm{H}^*(G, k) \cong \mathrm{Ext}^*_{kG}(k, k)$. Let $J(M)$ be the annihilator of $\mathrm{Ext}^*_{kG}(M, M)$ in $\mathrm{H}^*(G, k)$.

Let $V_G(k) = \mathrm{Proj}(\mathrm{H}^*(G, k))$ be the spectrum of homogeneous prime ideals in $\mathrm{H}^*(G, k)$.

Let $V_G(M)$ be the variety of $J(M)$, the set of all prime ideals that contain $J(M)$.

Proposition: Suppose that X is virtually M-projective. Then $V_G(X) \subseteq V_G(M)$.

Theorem: (Benson-C-Rickard) If $V_G(M) = V_G(k)$ then M generates **stmod**(kG) in the sense that every module in the category is a direct summand of some sequence of extensions of the modules $\Omega^n(M)$.

Question: Suppose that M has the property that $V_G(M) = V_G(k)$ (so that M generates $\mathbf{stmod}(kG)$). Is it necessary that k is virtually M-projective?

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Answer: No!

Well, not in general. Assume $p > 2$. There exist technical criteria for such a situation in the case that G is elementary abelian of order p^2. Such module inflate to modules that also imply a "no" for larger elementary abelian p-groups.

Proposition: Suppose that k is virtually M-projective of degree d. Let H be a subgroup of G, and assume that $\mathrm{H}^*(H, k)$ is generated as a module over $R = \mathrm{res}_{G,H}(\mathrm{H}^*(G, k))$ by elements in degree at most n. Then k_H is virtually $M_{\downarrow H}$-projective in degree at most $d + n$.

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Proposition: Suppose that X is virtually L-projective of degree d and virtually M-projective of degree e. Let n be a number such that the ring $\mathrm{Ext}^*_{kG}(X,X)$ is generated as a k-algebra in degrees at most n. Then X is vitually $L \otimes M$-projective of degree at most $d+e+n$.

Definition: An additive tensor ideal $\mathcal{C}$ is a full subcategory that is closed under arbitrary tensor products (meaning that if X is in $\mathcal{C}$, then $X \otimes Y$ is in $\mathcal{C}$ for any object Y) and finite direct sums and direct summands.

Note that an additive tensor ideal is not assumed to be triangulated - that is, not closed under extensions.

Example: We say a module is p-divisible if its dimension is divisible by p. (Or the long version is that M is p-divisible if all of its direct summands after any field extension have dimension divisible by p.) The collection of p-divisible modules is an additive tensor ideal. In fact, it is a prime ATI.

Example: Fix a natrual number n. The full subcategory of all M such that k is not virtually M-projective of degree more than n is an additive tensor ideal.

The *radical* of an additive category $\mathcal{C}$ is the ideal of morphisms

$$\mathrm{Rad}_{\mathcal{C}}(M,N) = \{f : M \to N | \text{ for all } g : N \to M,\ 1_M - gf \text{ is invertible}\}.$$

When $M = N$, the ideal $\mathrm{Rad}_{\mathcal{C}}(M,M)$ is the Jacobson radical of the ring $\mathrm{End}_{\mathcal{C}}(M)$.

Theorem (Balmer-C.) The tensor closure of the radical of **stmod**(kG) is the additive ideal given for indecomposable modules M and N by the formula:

1. If $M \not\cong N$, then $\mathcal{I}(M,N) := \mathrm{Hom}_{\mathcal{C}}(M,N)$.
2. If M is p-divisible, then $\mathcal{I}(M,M) := \mathrm{Hom}_{\mathcal{C}}(M,M)$.
3. If M is not p-divisible, then $\mathcal{I}(M,M) := \mathrm{Rad}_{\mathcal{C}}(M,M)$.

Corollary The Kelly radical of **stmod**(kG) is a tensor ideal if and only if G is cyclic of order p.

Suppose that H is a subgroup of G and we consider the relative stable category, $\mathbf{stmod}_H(kG)$, where the morphisms between objects are the standard homomorphisms modulo those that factor through an H-projective module (H-projective means induced from H). This is a triangulated category with triangles corresponding to exact sequence that are split on restriction to H.

Theorem: Suppose that $J \subseteq H$ and that $\mathcal{C}$ is an ATI in $\mathbf{stmod}(kJ)$. Let $\mathcal{L}$ be the full subcategory of $\mathbf{stmod}_H(kG)$ of all modules whose restriction to J is in $\mathcal{C}$. Then $\mathcal{L}$ is a thick subcategory of $\mathbf{stmod}_H(kG)$ and it is prime if $\mathcal{C}$ is prime.

Thanks!