A first look at homotopy dimer algebras on surfaces with boundary

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Conference on Geometric Methods in Representation Theory
University of Iowa
November 2017

Dimer quivers with boundary

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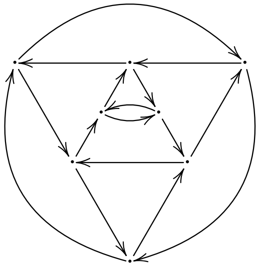
- A *boundary* of Q is a set $\mathcal{B}$ of connected components of $M \setminus Q$.
- A *$\mathcal{B}$-perfect matching* D is a set of arrows such that each unit cycle, which is not the boundary of a component in $\mathcal{B}$, contains precisely one arrow in D.
  Denote by $\mathcal{P}_{\mathcal{B}}$ the set of $\mathcal{B}$-perfect matchings.

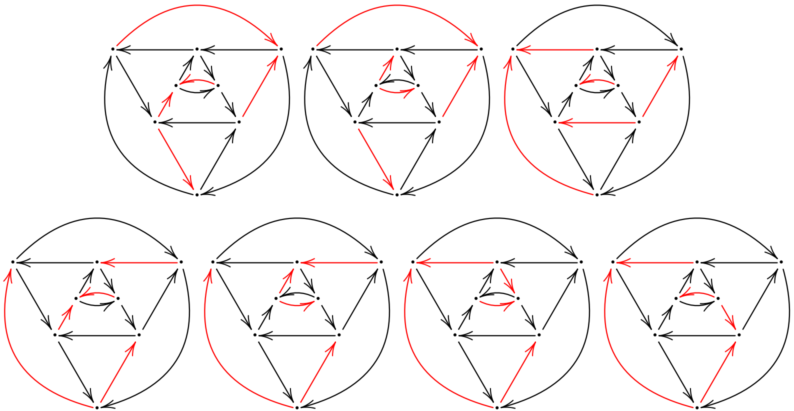
An example

Let Q be the quiver on the sphere S^2,



The outermost cycle of Q is a unit cycle since Q is on S^2.
Let $\mathcal{B}$ consist of the two faces bounded by the innermost and outermost unit cycles.

14 perfect matchings:



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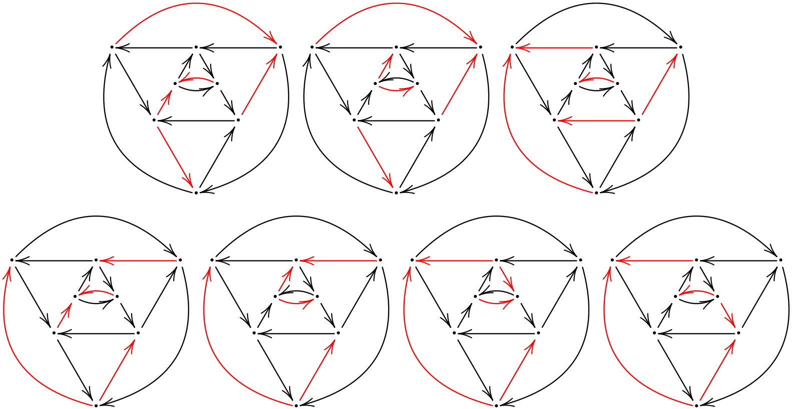
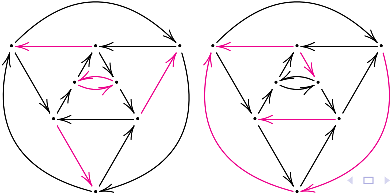


4 boundary perfect matchings:



Homotopy algebras with boundary

Consider the algebra homomorphism

$$\tau : kQ \to M_{|Q_0|}(k[x_D \mid D \in \mathcal{P}_{\mathcal{B}}])$$

defined on the vertices $e_i \in Q_0$ and arrows $a \in Q_1$ by

$$e_i \mapsto e_{ii} \quad \text{and} \quad a \mapsto \prod_{a \in D \in \mathcal{P}_{\mathcal{B}}} x_D \cdot e_{\mathrm{h}(a),\mathrm{t}(a)},$$

and extended multiplicatively to paths and k-linearly to kQ. The *homotopy algebra* of Q with boundary $\mathcal{B}$ is then the quotient

$$A := kQ / \ker \tau.$$

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We can view A as a tiled matrix algebra by identifying A with its image in $M_{|Q_0|}(k[x_D])$.
In our example, $A \subset M_8\left(k[x_1, \ldots, x_{18}]\right)$.

Let B be an integral domain and a k-algebra. Let

$$A = \left[A^{ij}\right] \subset M_d(B)$$

be a tiled matrix algebra; that is, each diagonal entry $A^i := A^{ii}$ is a unital subalgebra of B.

Definition

Set

$$R := k\left[\cap_{i=1}^{d} A^i\right] \quad \text{and} \quad S := k\left[\cup_{i=1}^{d} A^i\right].$$

We call S the *cycle algebra* of A.

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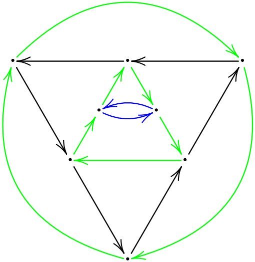
Proposition

The center of a homotopy algebra A is R.

Consider the cycles:



Let α, β, σ be the single nonzero matrix entries of the τ-images of the green, blue, and unit cycles respectively.
Then

$$\begin{aligned} S &= k[\alpha, \beta, \sigma]/(\alpha\beta - \sigma^2), \\ R &= k[\alpha, \sigma] + (\alpha, \sigma^2)S. \end{aligned}$$

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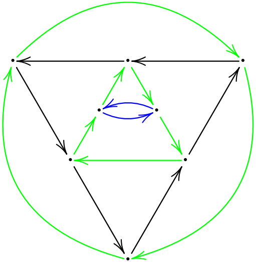

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$\Longrightarrow$ R is nonnoetherian and $R \neq S$ ...coincidence?

Noetherianity criteria

Theorem

Let A be a homotopy algebra with center R. Suppose there are monomials in S which are relatively prime in $k[x_D]$.
TFAE:

1. *Each arrow annihilates a simple A-module of dimension 1^{Q_0}.*
2. *A is a dimer algebra (i.e., the relations come from a potential).*
3. *$R = S$ (i.e., $A^i = A^j$ for each $i, j \in Q_0$).*
4. *A is noetherian.*
5. *R is noetherian.*
6. *A is a finitely generated R-module.*

Local endomorphism ring structure

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$$A_{\mathfrak{q}} := \left\langle \begin{bmatrix} A^1_{\mathfrak{q}\cap A^1} & A^{12} & \cdots & A^{1d} \\ A^{21} & A^2_{\mathfrak{q}\cap A^2} & & \\ \vdots & & \ddots & \\ A^{d1} & & & A^d_{\mathfrak{q}\cap A^d} \end{bmatrix} \right\rangle \subset M_d(\operatorname{Frac} B).$$

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- The *residue module* of A at $\mathfrak{q}$ is the left $A_{\mathfrak{q}}$-module,

$$A_{\mathfrak{q}}/\mathfrak{q} := \bigoplus_{1 \leq i \leq d} A_{\mathfrak{q}} e_i / \left(\mathfrak{q} \cap e_i A_{\mathfrak{q}} e_i\right).$$

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Conjecture

If $\mathfrak{q} \in \operatorname{Spec} S$ *is minimal over* $\mathfrak{q} \cap R$, *then* $A_{\mathfrak{q}}/\mathfrak{q}$ *is semi-simple,*

$$A_{\mathfrak{q}}/\mathfrak{q} \cong \bigoplus_{V \in \mathcal{S}_{\mathfrak{q}}} V,$$

where $\mathcal{S}_{\mathfrak{q}}$ *is the set of all simple* $A_{\mathfrak{q}}$*-modules, up to isomorphism.*

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- For each $V \in \mathcal{S}_{\mathfrak{q}}$, the *simple idempotent* corresponding to V is

$$\epsilon_V := \sum_{i \in Q_0 \,:\, e_i V \neq 0} e_i.$$

By our conjecture,

$$\sum_{V \in \mathcal{S}_{\mathfrak{q}}} \epsilon_V = 1_A.$$

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- For each boundary component b, let ε_b be the sum of the vertex idempotents around b.

Conjecture

1. *Let* $\mathfrak{q} \in \operatorname{Spec} S$ *be minimal over* $\mathfrak{q} \cap R$*, let* $V \in \mathcal{S}_{\mathfrak{q}}$*, and set* $\epsilon := \epsilon_V$*. Then for each* $i \in Q_0$ *satisfying* $e_i \epsilon \neq 0$*, we have*

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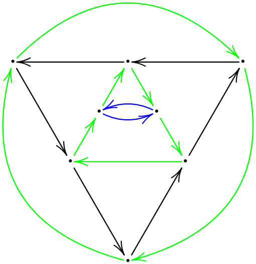
Proposition

Both conjectures hold for our example.

Recall:



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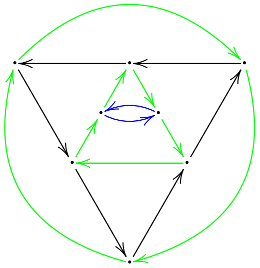

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The two boundary components correspond to the two prime ideals,

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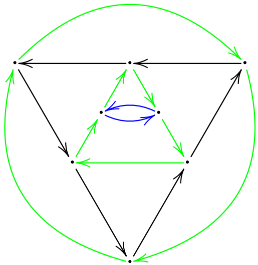

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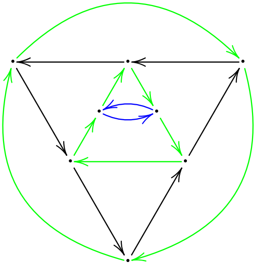

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