Prime Spectra of 2-Categories
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Overview

1. Category theory

2. The prime spectra

3. Applications to Richardson varieties
2-Categories

Definition

A 2-category is a category enriched over the category of small categories.

So a 2-category $\mathcal{T}$ has:

- Objects, denoted by $A_1, A_2$ etc;
- 1-morphisms between objects, denoted $f, g, h$, etc; set of 1-morphisms from $A_1$ to $A_2$ denoted $\mathcal{T}(A_1, A_2)$;
- 2-morphisms between 1-morphisms, denoted $\alpha, \beta, \gamma$, etc; set of 2-morphisms from $f$ to $g$ denoted $\mathcal{T}(f, g)$. 

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Kent Vashaw
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2-Categories

Composition of 1-morphisms:

\[ A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3. \]

Vertical composition of 2-morphisms \( \alpha \circ \beta \):

\[
\begin{array}{ccc}
A_1 & \xrightarrow{f} & A_2 \\
\downarrow^{\alpha} & & \downarrow^{\beta} \\
A_1 & \xrightarrow{g} & A_2 \\
\downarrow^{h} & & \\
A_2 & \xrightarrow{f} & A_3
\end{array}
\]

Horizontal composition of 2-morphisms \( \alpha_2 \ast \alpha_1 \):

\[
\begin{array}{ccc}
A_1 & \xrightarrow{f_1} & A_2 \\
\downarrow^{\alpha_1} & & \downarrow^{\alpha_2} \\
A_1 & \xrightarrow{g_1} & A_2 \\
\downarrow^{g_2} & & \\
A_1 & \xrightarrow{f_2} & A_3
\end{array}
\]
2-Categories

\((\alpha_1 \circ \beta_1) \ast (\alpha_2 \circ \beta_2) = (\alpha_1 \ast \alpha_2) \circ (\beta_1 \ast \beta_2)\):
Exact categories

Definition

A 1-category is called **exact** if:

- It is additive;
- It has a set of distinguished short exact sequences

\[ A_1 \rightarrow A_2 \rightarrow A_3 \]

that obey some axioms.
Exact categories

Some exact 1-categories:

- An additive category with short exact sequences defined by
  \[ A_1 \to A_1 \oplus A_3 \to A_3; \]

- Abelian categories with traditional short exact sequences
  \( (\ker g \cong \text{im } f); \)

- Full subcategories of abelian categories closed under extension.

Definition

A 2-category \( \mathcal{T} \) is exact if each set \( \mathcal{T}(A, B) \) is itself an exact 1-category.
Definition

Suppose $\mathcal{C}$ is an exact 1-category. Then the **Grothendieck group** of $\mathcal{C}$, denoted $K_0(\mathcal{C})$, is defined by:

- Take the free abelian group on objects of $\mathcal{C}$;
- For every exact sequence

$$0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0,$$

*quotient by the relation* $[A_1] + [A_3] = [A_2]$. 

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**Category theory**

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**Applications to Richardson varieties**
Definition

Suppose $\mathcal{T}$ is an exact 2-category. Then the **Grothendieck group** of $\mathcal{T}$, denoted $K_0(\mathcal{T})$ is defined as the 1-category with:

- **Objects** the same as $\mathcal{T}$;
- **Set of morphisms** from $X$ to $Y$ given by $K_0(\mathcal{T}(X, Y))$, the Grothendieck group of the 1-category $\mathcal{T}(X, Y)$.
- **Composition of morphisms** induced from composition of morphisms in $\mathcal{T}$. 
Positive part of the Grothendieck group

**Definition**

The **positive part of the Grothendieck group** of an exact 1-category $\mathcal{C}$, denoted $K_0(\mathcal{C})_+$, is defined as the subset of $K_0(\mathcal{C})$ forming a monoid under addition generated by the indecomposable objects.

In other words, while the Grothendieck group has all elements of the form

$$\sum_i \lambda_i [b_i], \lambda_i \in \mathbb{Z},$$

the positive part of the Grothendieck group has elements of the form

$$\sum_i \lambda_i [b_i], \lambda_i \in \mathbb{N}.$$
Definition

The positive part of the Grothendieck group of an exact 2-category $\mathcal{T}$, denoted $K_0(\mathcal{T})_+$, has the same objects as $\mathcal{T}$, with hom spaces $K_0(\mathcal{T})_+(X, Y)$ defined by $K_0(\mathcal{T}(X, Y))_+$. 
Strong categorification

- Let $A$ an algebra with orthogonal idempotents $e_i$ with $1 = e_1 + e_2 + ... + e_n$.
- $A = \bigoplus e_i A e_j$.
- Consider $A$ as a category: an object for each $e_i$, set of morphisms from $i$ to $j$ given by $e_i A e_j$.
- Composition of morphisms given by multiplication.
Strong categorification

\[ K_0(T) \xrightarrow{\text{view as an algebra}} A \]
Strong categorification

Definition

We call $B_+$ a $\mathbb{Z}_+\text{-ring}$ if $B_+$ has a basis (as a monoid) $\{b_i\}$ with relations $b_ib_j = \sum m_{i,j}^k b_k$ where all coefficients are positive. Elements are all positive linear combinations of basis elements, multiplication is extended from basis elements.

So we can view Grothendieck groups of 2-categories as $\mathbb{Z}$-algebras, and positive Grothendieck groups as $\mathbb{Z}_+$-rings.
Ideals

Definition

Let $\mathcal{T}$ be an exact 2-category where composition of 1-morphisms is an exact bifunctor. We call $\mathcal{I}$ a thick ideal of $\mathcal{T}$ if:

- $\mathcal{I}$ is a full subcategory of $\mathcal{T}$ such that if in $\mathcal{T}(X,Y)$ we have an exact sequence of 1-morphisms
  \[ 0 \to f_1 \to f_2 \to f_3 \to 0, \]
  then $f_2$ is in $\mathcal{I}$ if and only if $f_1$ and $f_2$ are in $\mathcal{I}$;

- $\mathcal{I}$ is an ideal: if $f \in (X,Y)$ is in $\mathcal{I}$ and $g \in \mathcal{T}(Y,Z)$ then $g \circ f \in \mathcal{I}$; and if $h \in \mathcal{T}(W,X)$ then $f \circ h \in \mathcal{I}$. 
Ideals

**Definition**

Suppose \( \mathcal{M} \) is any subset of 1-morphisms and 2-morphisms of a 2-category \( \mathcal{T} \). Then we define the **thick ideal generated by** \( \mathcal{M} \), denoted \( \langle \mathcal{M} \rangle \), to be the smallest thick ideal that contains \( \mathcal{M} \), which is the intersection of all thick ideals containing \( \mathcal{M} \).

**Definition**

Suppose \( B_+ \) is a \( \mathbb{Z}_+ \)-ring. Then \( I \subset B_+ \) is a **thick ideal** if \( a + b \) is in \( I \) if and only if \( a \) and \( b \) are in \( I \), and we also have that if \( i \) is in \( I \), then \( ai \) and \( ia \) are in \( I \) for every \( a \in B_+ \).
Prime Spectra of 2-Categories

Kent Vashaw

Category theory

The prime spectra

Applications to Richardson varieties

Prime and completely prime ideals

Definition

We call \( \mathcal{P} \) a **prime** of \( \mathcal{T} \) if \( \mathcal{P} \) is a thick ideal of \( \mathcal{T} \) such that if \( \mathcal{I} \) and \( \mathcal{J} \) are thick ideals in \( \mathcal{T} \), then if \( \mathcal{I} \circ \mathcal{J} \subset \mathcal{P} \), then either \( \mathcal{I} \subset \mathcal{P} \) or \( \mathcal{J} \subset \mathcal{P} \). We call \( \mathcal{I} \) **completely prime** if it is a thick ideal such that \( f \circ g \in \mathcal{I} \) implies either \( f \in \mathcal{I} \) or \( g \in \mathcal{I} \).

Definition

The set of all primes \( \mathcal{P} \) of a 2-category \( \mathcal{T} \) is called the **spectrum** of \( \mathcal{T} \) and is denoted \( \text{Spec}(\mathcal{T}) \).
Definition

Suppose $B_+$ is a $\mathbb{Z}_+$-ring. Then we call $P$ a prime if $P$ is a thick ideal, and $IJ \subseteq P$ implies $I$ or $J$ is in $P$ for all thick ideals $I$ and $J$. 
General results

We obtain many results with respect to $\text{Spec}(\mathcal{T})$ that correspond to the prime spectra of noncommutative rings.

**Theorem**

A thick ideal $\mathcal{P}$ is prime if and only if: for all 1-morphisms $m, n$ of $\mathcal{T}$ with $m \circ \mathcal{T} \circ n \in \mathcal{P}$, either $m \in \mathcal{P}$ or $n \in \mathcal{P}$.

This corresponds to the result in the classical theory:

**Theorem**

An ideal $P$ of a ring $R$ is prime if and only if: for all $x, y \in R$, if $xRy \subset P$ then $x$ or $y$ is in $P$. 
General results

Theorem

A thick ideal $\mathcal{P}$ is prime if and only if: for all thick ideals $\mathcal{I}, \mathcal{J}$ properly containing $\mathcal{P}$, we have that $\mathcal{I} \circ \mathcal{J} \not\subset \mathcal{P}$.

Theorem

Every maximal thick ideal is prime.

Theorem

The spectrum of an exact 2-category $\mathcal{T}$ is nonempty.
There is a bijection between \( \text{Spec}(\mathcal{T}) \) and \( \text{Spec}(K_0(\mathcal{T})_+) \).

Let \( \mathcal{T} \) be a categorification of \( A \). Consider the map \( \phi : \text{Spec}(K_0(\mathcal{T})_+) \to \text{Ideals}(K_0(\mathcal{T})) = A \) defined by
\[
\phi(P) = \{ x - y : x, y \in P \}.
\]

In general, \( \phi \) is not a map \( \text{Spec}(K_0(\mathcal{T})_+) \to \text{Spec}(K_0(\mathcal{T})) \).

Example: let \( H \) be a Hopf algebra, \( \mathcal{T} \) be the category of finitely generated \( H \)-modules. Then \( \{0\} \) is completely prime in \( K_0(\mathcal{T})_+ \) but not in \( K_0(\mathcal{T}) \).
Relationship between the spectra

![Diagram]

**Lemma**

Let $\mathcal{T}$ be a categorification of $A$. If $\phi(P)$ is a prime in $K_0(\mathcal{T})$, and $\mathcal{P}$ is the prime in $\mathcal{T}$ corresponding to $P$, then $A/\phi(P)$ is categorified by the Serre quotient $\mathcal{T}/\mathcal{P}$. 
Coordinate rings of Richardson varieties

Definition

Suppose $G$ is a connected simple Lie group, $B_\pm$ opposite Borel subgroups, and $W$ the Weyl group. Then the Richardson variety of $u$ and $w \in W$ is

$$R_{u,w} = B_- \cdot uB_+ \cap B_+ \cdot wB_+ \subset G/B_+.$$  

Individually, $B_- \cdot uB_+$ and $B_+ \cdot wB_+$ are called Schubert cells.
Coordinate rings of Richardson varieties

Theorem (Yakimov)

\[ G/B_+ = \bigsqcup_{u \leq w, u, w \in W} R_{u,w}. \]

Applications of Richardson varieties:

- Representation theory (Richardson, Kazhdan, Lusztig, Postnikov);
- Total positivity (Lusztig);
- Poisson geometry (Brown, Goodearl, and Yakimov);
- Algebraic geometry (Knutson, Lam, Speyer);
- Cluster algebras (Leclerc).
Coordinate rings of Richardson varieties

We restrict to $u = 1$ for simplicity.
Let $U_q(n_+)$ denote the subset of $U_q(g)$ generated by the $E_i$ Chevalley generators.

**Theorem (Yakimov)**

If $T$ is a maximal torus of $G$, then $T$ acts on $U_q(n_+)$ via an algebra automorphism. The $T$-invariant prime ideals are parametrized by elements of $W$.

**Theorem (Yakimov)**

$U_q(n_+)/I_w$ is a quantization of the coordinate ring $\mathbb{C}[R_{1,w}]$. 
Coordinate rings of Richardson varieties

We want to produce a categorification of $U_q(n_+)/I_w$.

**Theorem (Khovanov and Lauda)**

There exists a categorification $\mathcal{U}^+$ of $U_q(n_+)$ that is a tensor category of modules of KLR-algebras.
We are currently working on showing that $I_w$ is a prime in $\text{Spec}(U_q(n_+))$ corresponding to a prime in $\text{Spec}(U_q(n_+)_+).$ Then if $I_w$ is the prime in $\text{Spec}(\mathcal{U}^+)$ corresponding to $I_w$, then

$$\mathcal{U}^+/I_w$$

will categorify quantization of the coordinate ring of the Richardson variety.
Conclusion

Thanks for listening!