Universal Deformation Rings: Semidihedral and Generalized Quaternion 2-groups

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Joint Work with Frauke Bleher and Ted Chinburg

Introduction

Question

Let k be an algebraically closed field of prime characteristic p. Let G be a finite group and V a finitely generated kG-module.

When can V be lifted to a module for G over a complete discrete valuation ring, such as the ring of infinite Witt vectors $W = W(k)$ over k?

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Examples

1. If all 2-extensions of V by itself are trivial, then V can always be lifted over W (Green, 1959).
2. Every endo-trivial kG-module can be lifted to an endo-trivial WG-module (Alperin, 2001).

Goals

Definition

For $n \geq 4$, let SD_n denote the semidihedral group of order 2^n, i.e.,

$$SD_n = \langle x, y | x^{2^{n-1}} = y^2 = 1, yxy^{-1} = x^{2^{n-2}-1} \rangle.$$

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Definition

For $n \geq 3$, let GQ_n denote the (generalized) quaternion group of order 2^n, i.e.,

$$GQ_n = \langle x, y | x^{2^{n-1}} = 1, x^{2^{n-2}} = y^2, yxy^{-1} = x^{-1} \rangle.$$

Main Result

Proposition (Bleher, Chinburg, S)

Let k be an algebraically closed field of characteristic 2, let W be the ring of infinite Witt vectors over k, and let $D = SD_n$ or $D = GQ_n$. Then if V is a finitely generated endo-trivial kD-module we have the following:

1) $R(D, V) \cong W[\mathbb{Z}/2 \times \mathbb{Z}/2]$ *and*
2) *Every universal lift U of V over $R = R(D, V)$ is endo-trivial in the sense that the $U^* \otimes_R U \cong R \oplus Q_R$, as RD-modules, where Q_R is a free RD-module.*

General setup

Let k be an algebraically closed field of prime characteristic p, and let $W = W(k)$ be the ring of infinite Witt vectors over k.

Let $\mathcal{C}$ be the category of all complete local commutative Noetherian rings R with residue field k, where the morphisms are local homomorphisms of local rings which induce the identity on the residue field k.

Note that all rings R in $\mathcal{C}$ have a natural W-algebra structure, meaning that the morphisms in $\mathcal{C}$ can also be viewed as continuous W-algebra homomorphisms inducing the identity on k.

Let G be a finite group, let V be a finitely generated kG-module, and let R be an object in $\mathcal{C}$.

Deformations

Definition

(i) A *lift* of V over R is a pair, (M, ϕ), *where*

- M is a finitely generated RG-module, that is free over R.
- $\phi : k \otimes_R M \longrightarrow V$ is a kG-module isomorphism.

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(ii) $(M, \phi) \cong (M', \phi')$ as lifts, if there exists an RG-module isomorphism $f : M \longrightarrow M'$ such that the following diagram commutes

$$\begin{array}{ccc} k \otimes_R M & \xrightarrow{\mathrm{id} \otimes f} & k \otimes_R M' \\ & {\scriptstyle \phi} \searrow \quad \swarrow {\scriptstyle \phi'} & \\ & V & \end{array}$$

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(iii) Let $[M, \phi]$ denote the isomorphism class of a lift (M, ϕ) of V over R. This isomorphism class is called a *deformation* of V over R.

Universal deformation rings

Definition

Suppose there exists a ring $R(G, V)$ in $\mathcal{C}$ and a lift $(U(G, V), \phi_U)$ of V over $R(G, V)$ such that for all rings R in $\mathcal{C}$ and for each lift (M, ϕ) of V over R there exists a unique morphism

$$\alpha : R(G, V) \to R$$

in $\mathcal{C}$ such that

$$(M, \phi) \cong (R \otimes_{R(G,V),\alpha} U(G, V), \phi'_U)$$

where ϕ'_U is the composition

$$k \otimes_R (R \otimes_{R(G,V),\alpha} U(G, V)) \cong k \otimes_{R(G,V)} U(G, V) \xrightarrow{\phi} V \, .$$

Then $R(G, V)$ is called the universal deformation ring of V, and $[U(G, V), \phi_U]$ is called the universal deformation of V.

Modules with stable endomorphism ring k

Theorem (Bleher and Chinburg, 2000)

Let V be a finitely generated kG-module such that

$$\underline{\mathrm{End}}_{kG}(V) \cong k.$$

Then

(i) *V has a universal deformation ring $R(G, V)$,*

(ii) *$R(G, \Omega(V)) \cong R(G, V)$, and*

(iii) *there exists a non-projective indecomposable kG-module V_0 such that*

- $\underline{\mathrm{End}}_{kG}(V_0) \cong k$,
- *$V \cong V_0 \oplus Q$ for some projective kG-module Q, and*
- $R(G, V) \cong R(G, V_0)$.

Endo-trivial kSD_n-modules

Summary (Carlson and Thévenaz, 2000)

Let k be an algebraically closed field of characteristic 2 and let $z = x^{2^{n-2}}$, and let

$$H = \langle x^{2^{n-3}}, yx \rangle, E = \langle y, z \rangle.$$

Let $T(SD_n)$ denote the group of equivalence classes of endo-trivial kSD_n-modules and consider the restriction map

$$\Xi_{SD_n} : T(SD_n) \to T(E) \times T(H) \cong \mathbb{Z} \times \mathbb{Z}/4.$$

Then Ξ_{SD_n} is injective, $T(SD_n) \cong \mathbb{Z} \times \mathbb{Z}/2$, and $T(SD_n)$ is generated by $[\Omega^1_{SD_n}(k)]$ and $[\Omega^1_{SD_n}(L)]$, where

$$Y = k[SD_n/\langle y \rangle] \text{ and } L = \text{rad}(Y).$$

A different point of view

Lemma

Let $\Lambda_{SD_n} = k\langle a, b\rangle / I_{SD_n}$, *where*

$$I_{SD_n} = \Big((ab)^{2^{n-2}} - (ba)^{2^{n-2}}, a^2 - b(ab)^{2^{n-2}-1} - (ab)^{2^{n-2}-1}, \\ b^2, (ab)^{2^{n-2}} a \Big)$$

Let $z = x^{2^{n-2}}$ *and define* $r_a, r_b \in \mathrm{rad}(kSD_n)$ *by*

$$r_a = (z + yx) + (x + x^{-1}) + \sum_{i=1}^{2^{n-4}-1} (x^{4i+1} + x^{-(4i+1)})(1 + zy)$$

$$r_b = 1 + y$$

Then the map (Bondarenko and Drozd, 1977) $f_{SD_n} : \Lambda_{SD_n} \to kSD_n$ *defined by*

$$f_{SD_n}(a) = r_a, f_{SD_n}(b) = r_b$$

induces a k*-algebra isomorphism.*

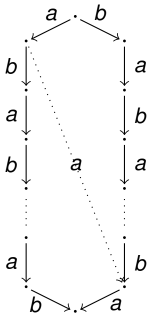
A different point of view

Lemma

Let $\Lambda = \Lambda_{SD_n}$ and define the following Λ-modules

$$Y_\Lambda = \Lambda b \text{ and } L_a = \Lambda ab.$$

Then $Y_\Lambda \cong \Lambda/\Lambda b$ and $L_a \cong \Lambda a/\Lambda a^2 \cong \Lambda/\Lambda a$. Moreover, Y_Λ and L_a are uniserial Λ-modules of length 2^{n-1} and $2^{n-1} - 1$, respectively. Furthermore, $f_{SD_n}(Y_\Lambda) = Y$ and $f_{SD_n}(L_a) = L$.



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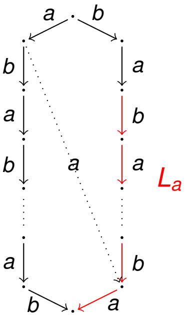


The component of the stable AR-quiver $\Gamma_S(kSD_n)$ containing L

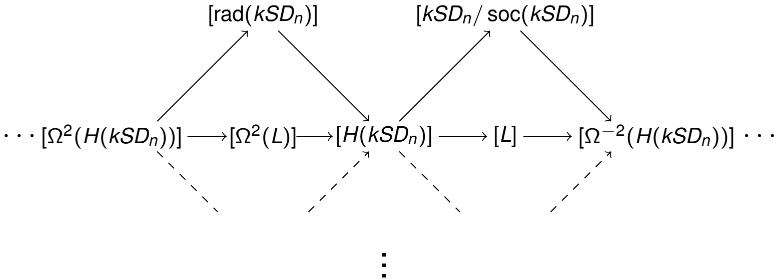


Figure: A consequence of Erdmann's work

Endo-trivial kGQ_n-modules

Summary (Carlson and Thévenaz, 2000)

Let $T(GQ_n)$ denote the group of equivalence classes of endo-trivial kGQ_n-modules. Then there exists an endo-trivial kGQ_n-module L with k-dimension $2^{n-1}-1$. If $n=3$, then $T(GQ_n) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$ generated by $[\Omega^1_{GQ_n}(k)]$ and $[\Omega^1_{GQ_n}(L)]$. If $n \geq 4$ then let

$$H = \langle yx, x^{2^{n-3}} \rangle, H' = \langle y, x^{2^{n-3}} \rangle$$

and consider the restriction map

$$\Xi_{GQ_n} : T(GQ_n) \to T(H) \times T(H') \cong \mathbb{Z}/4 \times \mathbb{Z}/4.$$

Then Ξ_{GQ_n} is injective, $T(GQ_n) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$, and $T(GQ_n)$ is generated by $[\Omega^1_{GQ_n}(k)]$ and $[\Omega^1_{GQ_n}(L)]$. Moreover, for all $n \geq 3$ we have that

$$T(GQ_n) = \{[\Omega^i_{GQ_n}(k)]\}_{i=0}^3 \cup \{[\Omega^i_{GQ_n}(L)]\}_{i=0}^3.$$

A different point of view

Lemma

Let $\Lambda_{GQ_n} = k\langle a, b\rangle / I_{GQ_n}$, *where*

$$I_{SD_n} = \Big((ab)^{2^{n-2}} - (ba)^{2^{n-2}}, a^2 - b(ab)^{2^{n-2}-1} - \delta(ab)^{2^{n-2}},$$
$$b^2 - a(ba)^{2^{n-2}-1} - \delta(ab)^{2^{n-2}}, (ab)^{2^{n-2}}a\Big) \text{ and}$$

$$\delta = \begin{cases} 0 \text{ if } n = 3 \\ 1 \text{ if } n \geq 4 \end{cases}$$

If $n = 3$, *let* ω *be a primitive cube root of unity in* k *and define* $r_a, r_b \in \mathrm{rad}(kSD_n)$ *by*

$$\begin{aligned} r_a =& (1 + x) + \omega(1 + yx) + \omega^2(1 + y) \\ r_b =& (1 + x) + \omega^2(1 + yx) + \omega(1 + y) \end{aligned}$$

A different point of view

Lemma (Continued)

If $n \geq 4$, define $r, r_a, r_b \in \mathrm{rad}(kSD_n)$ as follows

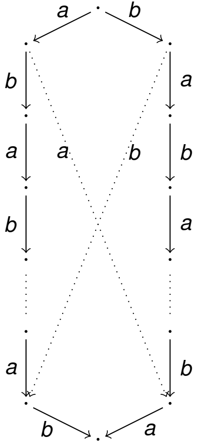
$$
\begin{aligned}
r =& (yx + y)^{2^{n-1}-3} + \sum_{i=1}^{n-3} (yx + y)^{2^{n-2}-2^i}, \\
r_a =& (1 + yx + r) + [(1 + yx + r)(1 + y + r)]^{2^{n-2}-1}, \\
r_b =& (1 + y + r) + [(1 + yx + r)(1 + y + r)]^{2^{n-2}-1}
\end{aligned}
$$

Then the map (Dade, 1972) $f_{GQ_n} : \Lambda_{GQ_n} \to kGQ_n$ defined by

$$f_{GQ_n}(a) = r_a, f_{GQ_n}(b) = r_b$$

induces a k-algebra isomorphism.

A visualization of kGQ_n



A different point of view

Lemma

Let $\Lambda = \Lambda_{GQ_n}$ *and define the following* Λ*-modules*

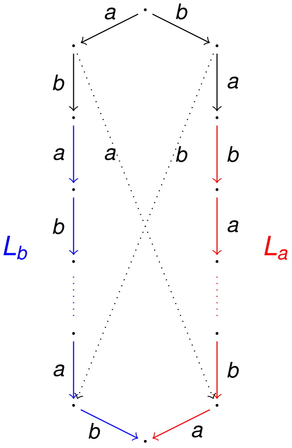
$$L_a = \Lambda ab \text{ and } L_b = \Lambda ba.$$

Then $L_a \cong \Lambda/\Lambda a$ *and* $L_b \cong \Lambda/\Lambda b$ *and both are uniserial of length* $2^{n-1} - 1$ *whose stable endomorphism rings are isomorphic to* k*. Moreover, the* Ω*-orbit of* L_a *is as follows:*

$$\Omega^1_\Lambda(L_a) \cong \Lambda a; \Omega^2_\Lambda(L_a) \cong L_b; \Omega^3_\Lambda(L_a) \cong \Lambda b; \Omega^4_\Lambda(L_a) \cong L_a,$$

and L_a *and* L_b *lie at the end of a 2-tube in the stable Auslander-Reiten quiver of* Λ*.*
Furthermore the endo-trivial kGQ_n*-module* L *corresponds under* f_{GQ_n} *to either* L_a *or* L_b*, and the* Ω*-orbit of* L *corresponds to the* Ω*-orbit of* L_a*.*

A visualization of L_a and L_b



Keys to proof

Proof outline

Let $D_n = SD_n$ or $D_n = GQ_n$, $n \geq 4$, and let $\Lambda = \Lambda_{D_n}$. Moreover, recall the isomorphism $f_{D_n} : \Lambda \to kD_n$ and the uniserial and endo-trivial kD_n-module $L_a = \Lambda ab$ which we will denote by L. We let $\rho \in \{yx, x\}$ and we denote $V = \Omega^{-1}(L)$ and let $R = R(D_n, V)$.

- Show that $\mathrm{Res}^{D_n}_{\langle\rho\rangle} V \cong k \oplus P_\rho$ where P_ρ is a free $k\langle\rho\rangle$-module.

- Note that $R(\langle\rho\rangle, \mathrm{Res}^{D_n}_{\langle\rho\rangle} V) \cong W[\langle\rho\rangle]$
  Thus we obtain a W-algebra homomorphism

$$\beta : W[\langle y\rangle] \otimes_W W[\langle yx\rangle] \to R.$$

- Determine the lifts of V to $k[\epsilon]/(\epsilon^2)$.

Keys to proof (cont.)

Proof outline (cont)

- Show that $\beta : W[\langle y \rangle] \otimes_W W[\langle yx \rangle] \to R$ is surjective
- Then show there exists a surjective W-algebra homomorphism

$$\alpha : W[\mathbb{Z}/2 \times \mathbb{Z}/2] \to R.$$

- Show that there exist four pairwise non-isomorphic lifts of V over W.
- Conclude that $R(SD_n, V) \cong W[\mathbb{Z}/2 \times \mathbb{Z}/2]$.

The case when $n = 3$

Proposition

Let V be a uniserial kQ_8-module of length 3 and let $R = R(Q_8, V)$ be its versal deformation ring. Let σ be the outer automorphism of order 3 such that σ cyclically permutes (x, y, yx).

i. *V is endo-trivial and R is a universal deformation ring of V.*

ii. $R/2R \cong k[[\mathbb{Z}/2 \times \mathbb{Z}/2]]$.

iii. *Twisting the action of Q_8 by σ induces a non-trivial k-linear transformation on the space of deformations of V over $k[\epsilon]$.*

The case when $n = 3$

Proposition (continued)

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 : R \to W$ be the four pairwise surjective morphisms in $\mathcal{C}$ corresponding to four non-isomorphic lifts of V over W obtained by twisting one particular lift of V over W by the four linear representations of Q_8 over W.

iv. *There exists an injective W-algebra homomorphism*

$$\alpha : R \to W \times W \times W \times W, \text{ given by } \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4).$$

v. *Twisting the action of Q_8 by σ induces a non-trivial automorphism β_σ of the universal deformation ring R in $\mathcal{C}$.*

Questions?

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Thank you!