# Isotropic Schur roots

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# Outline

- Describe the perpendicular category of an isotropic Schur root.
- Describe the ring of semi-invariants of an isotropic Schur root.
- Construct all isotropic Schur roots.

# Quivers, dimension vectors

- $k = \bar{k}$ is an algebraically closed field.
- $Q = (Q_0, Q_1)$ is an acyclic quiver with $Q_0 = \{1, 2, \ldots, n\}$.
- $\mathrm{rep}(Q)$ denotes the category of finite dimensional representations of $Q$ over $k$.
- Given $M \in \mathrm{rep}(Q)$, we denote by $\mathbf{d_M} \in (\mathbb{Z}_{\geq 0})^n$ its dimension vector.

# Bilinear form and roots

- We denote by $\langle -, - \rangle$ the Euler-Ringel form of $Q$.
- For $M, N \in \mathrm{rep}(Q)$, we have

$$\langle \mathbf{d_M}, \mathbf{d_N} \rangle = \dim_k \mathrm{Hom}(M, N) - \dim_k \mathrm{Ext}^1(M, N).$$

# Roots and Schur roots

- $\mathbf{d} \in (\mathbb{Z}_{\geq 0})^n$ is a (positive) root if $\mathbf{d} = \mathbf{d_M}$ for some indecomposable $M \in \mathrm{rep}(Q)$.
- Then $\langle \mathbf{d}, \mathbf{d} \rangle \leq 1$ and we call $\mathbf{d}$:

$$\begin{cases} \text{real}, & \text{if } \langle \mathbf{d}, \mathbf{d} \rangle = 1; \\ \text{isotropic}, & \text{if } \langle \mathbf{d}, \mathbf{d} \rangle = 0; \\ \text{imaginary}, & \text{if } \langle \mathbf{d}, \mathbf{d} \rangle < 0; \end{cases}$$

- A representation $M$ is Schur if $\mathrm{End}(M) = k$.
- If $M$ is a Schur representation, then $\mathbf{d_M}$ is a Schur root.
- We have real, isotropic and imaginary Schur roots.
- $\{$iso. classes of excep. repr.$\} \overset{1-1}{\longleftrightarrow} \{$real Schur roots$\}$.

# Perpendicular categories

- For $\mathbf{d}$ a dimension vector, we set $\mathcal{A}(\mathbf{d})$ the subcategory

$$\mathcal{A}(\mathbf{d}) = \{X \in \mathrm{rep}(Q) \mid \mathrm{Hom}(X, N) = 0 = \mathrm{Ext}^1(X, N)$$

$$\text{for some } \; N \in \mathrm{rep}(Q, \mathbf{d})\}.$$

- $\mathcal{A}(\mathbf{d})$ is an exact extension-closed abelian subcategory of $\mathrm{rep}(Q)$.
- If $V$ is rigid (in particular, exceptional), then $\mathcal{A}(\mathbf{d_V}) = {}^\perp V$.

**Proposition (-, Weyman)**

*For a dimension vector* $\mathbf{d}$, $\mathcal{A}(\mathbf{d})$ *is a module category* $\Leftrightarrow$ $\mathbf{d}$ *is the dimension vector of a rigid representation.*

# Perpendicular category of an isotropic Schur root

- Let $\boldsymbol{\delta}$ be an isotropic Schur root of $Q$ (so $\langle \boldsymbol{\delta}, \boldsymbol{\delta} \rangle = 0$).

**Proposition (-, Weyman)**

*There is an exceptional sequence* $(M_{n-2}, \ldots, M_1)$ *in* $\mathrm{rep}(Q)$ *where all* $M_i$ *are simples in* $\mathcal{A}(\boldsymbol{\delta})$.

- Complete this to a full exceptional sequence $(M_{n-2}, \ldots, M_1, V, W)$.

# Perpendicular category of an isotropic Schur root

- Starting with $(M_{n-2}, \ldots, M_1, V, W)$ and reflecting, we get an exceptional sequence
- $E := (M_{i_1}, M_{i_2}, \ldots, M_{i_r}, V', W', N_1, \ldots, N_{n-r-2})$.
- Consider $\mathcal{R}(Q, \delta) := \mathrm{Thick}(M_{i_1}, M_{i_2}, \ldots, M_{i_r}, V', W')$.

# Perpendicular category of an isotropic Schur root

**Theorem (-, Weyman)**

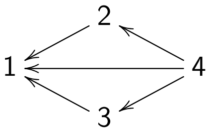
*The category $\mathcal{R}(Q,\delta)$ is tame connected with isotropic Schur root $\bar{\delta}$. It is uniquely determined by $(Q,\delta)$. The simple objects in $\mathcal{A}(\delta)$ are:*

- *The $M_i$ with $1 \leq i \leq n-2$,*
- *The quasi-simple objects of $\mathcal{R}(Q,\delta)$ (which includes some of the $M_i$).*

- In particular, the dimension vectors of those simple objects are either $\bar{\delta}$ or finitely many real Schur roots.

# An example

Consider the quiver




- We take $\boldsymbol{\delta} = (3, 2, 3, 1)$.
- We get an exceptional sequence whose dimension vectors are $((8, 3, 3, 3), (0, 0, 1, 0), (0, 1, 0, 0), (3, 3, 3, 1))$.
- We have $\boldsymbol{\delta} = (3, 3, 3, 1) - (0, 1, 0, 0)$.
- $\bar{\boldsymbol{\delta}} = (3, 2, 1, 1)$.
- Simple objects in $\mathcal{A}(\boldsymbol{\delta})$ are of dimension vectors $(0, 0, 1, 0), (8, 3, 3, 3)$ or $(3, 2, 1, 1)$.
- We have $\mathcal{R}(Q, \boldsymbol{\delta})$ of Kronecker type.

# An example

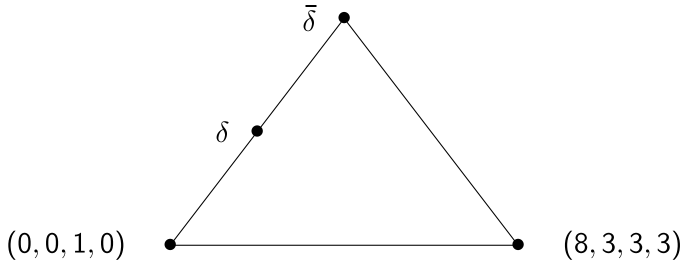



Figure : The cone of dimension vectors for $\boldsymbol{\delta} = (3,2,3,1)$

# Geometry of quivers

- For $\mathbf{d} = (d_1, \ldots, d_n)$ a dimension vector, denote by $\mathrm{rep}(Q, \mathbf{d})$ the set of representations $M$ with $M(i) = k^{d_i}$.
- $\mathrm{rep}(Q, \mathbf{d})$ is an affine space.
- For such a $\mathbf{d}$, we set $\mathrm{GL}(\mathbf{d}) = \prod_{1 \leq i \leq n} \mathrm{GL}_{d_i}(k)$.
- The group $\mathrm{GL}(\mathbf{d})$ acts on $\mathrm{rep}(Q, \mathbf{d})$ and for $M \in \mathrm{rep}(Q, \mathbf{d})$ a representation, $\mathrm{GL}(\mathbf{d}) \cdot M$ is its isomorphism class in $\mathrm{rep}(Q, \mathbf{d})$.

# Semi-invariants

- Take $\mathrm{SL}(\mathbf{d}) = \prod_{1 \leq i \leq n} \mathrm{SL}_{d_i}(k) \subset \mathrm{GL}(\mathbf{d})$.
- The ring $\mathrm{SI}(Q, \mathbf{d}) := k[\mathrm{rep}(Q, \mathbf{d})]^{\mathrm{SL}(\mathbf{d})}$ is the ring of semi-invariants of $Q$ of dimension vector $\mathbf{d}$.
- This ring is always finitely generated.

# Semi-invariants

- Given $X \in \mathrm{rep}(Q)$ with $\langle \mathbf{d_X}, \mathbf{d} \rangle = \mathbf{0}$, we can construct a semi-invariant $C^X(-)$ in $\mathrm{SI}(Q, \mathbf{d})$.
- We have that $C^X(-) \neq 0 \Leftrightarrow X \in \mathcal{A}(\mathbf{d})$.

**Proposition (Derksen-Weyman, Schofield-Van den Bergh)**

*These semi-invariants span* $\mathrm{SI}(Q, \mathbf{d})$ *over* $k$.

# Ring of semi-invariants of an isotropic Schur root

- The ring $\mathrm{SI}(Q, \mathbf{d})$ is generated by the $C^X(-)$ where $X$ is simple in $\mathcal{A}(\mathbf{d})$.

**Theorem (-, Weyman)**

*We have that $\mathrm{SI}(Q, \boldsymbol{\delta})$ is a polynomial ring over $\mathrm{SI}(\mathcal{R}, \bar{\boldsymbol{\delta}})$.*

**Corollary**

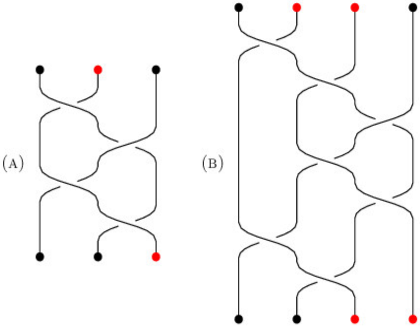
*By a result of Skowroński - Weyman, $\mathrm{SI}(Q, \boldsymbol{\delta})$ is a polynomial ring or a hypersurface.*

# Exceptional sequences of isotropic types

- A full exceptional sequence $E = (X_1, \ldots, X_n)$ is of isotropic type if there are $X_i, X_{i+1}$ such that $\mathrm{Thick}(X_i, X_{i+1})$ is tame.
- Isotropic position is $i$ and root type $\boldsymbol{\delta}_E$ is the unique iso. Schur root in $\mathrm{Thick}(X_i, X_{i+1})$.
- The braid group $B_n$ acts on full exceptional sequences.
- This induces an action of $B_{n-1}$ on exceptional sequences of isotropic type.

# An example

- Consider an exceptional sequence $E = (X, U, V, Y)$ of isotropic type with position 2.




- The exceptional sequence $E' = (X', Y', U', V')$ is of isotropic type with isotropic position 3.

# Constructing isotropic Schur roots

**Theorem (-, Weyman)**

*An orbit of exceptional sequences of isotropic type under $B_{n-1}$ always contains a sequence $E$ with $\delta_E$ an isotropic Schur root of a tame full subquiver of $Q$.*

**Corollary**

*There are finitely many orbits under $B_{n-1}$.*

- We can construct all isotropic Schur roots starting from the *easy ones*.

THANK YOU

Questions ?