The varieties of semi-conformal vectors of vertex operator algebras

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Conference on Geometric Methods in Representation Theory

University of Missouri

November 20, 2016

1. Motivations

• moduli problem in representation theory is to classify isomorphism classes of objects.

• The moduli problem always has the form of an algebraic variety X together with an algebraic group G acting on X. Hence the goal is to understand the invariants of "X/G" in the many different ways.

• This work is to apply the geometric ideas to theory of vertex operator algebras with influence of such ideology.

2. Vertex operator algebras

Vertex algebra (VA): $(V, Y^V, 1^V)$.

• V— a $\mathbb{C}$-vector space

• $Y^V : V \to \mathrm{End}_{\mathbb{C}}(V)[[z^{-1}, z]]$, (state-field corresp.)

$$v \mapsto Y^V(v,z) = \sum_n v_n z^{-n-1},\ v_n \in \mathrm{End}_{\mathbb{C}}(V)$$

such that $Y^V(V,z)V \subseteq V[z^{-1}, z]]$. Such $Y(v,z)$ are called *fields* and elements v in V are called *states*.

- (the *locality property*)

$(z_1 - z_2)^k [Y(u, z_1), Y(v, z_2)] = 0$ for some $k = k(v, u) > 0$

- $Y^V(\mathbf{1}, z) = Id$, and $Y^V(v, z)\mathbf{1} \in v + zV[[z]]$.
- there is a linear operator $D : V \to V$ such that

$$Y(D(v), z) = \frac{d}{dz} Y(v, z)$$

Remark: The locality together with the operator D implies the Jacobi identity which can be written as

$$\sum_{i=0}^{\infty} (-1)^i \binom{l}{i} (u_{m+l-i} v_{n+i} - (-1)^l v_{n+l-i} u_{m+i})$$
$$= \sum_{i=0}^{\infty} \binom{m}{i} (u_{l+i}(v))_{m+n-i}$$

for all $l, m, n \in \mathbb{Z}$.

Equivalently, taking $l = 0$,

$$[u_m, v_n] = \sum_{i=0}^{\infty} \binom{m}{i} (u_i(v))_{m+n-i}$$

for all $n, m \in \mathbb{Z}$.

Vertex operator algebra (VOA): $(V, Y^V, \mathbf{1}^V, \omega^V)$.

- $(V, Y^V, \mathbf{1}^V)$ is a vertex algebra and
- $\omega \in V$ such that for $Y^V(\omega, z) = \sum_n L(n) z^{-n-2}$ such

that the span of $\{L(n) : n \in \mathbb{Z}\}$ is a Lie algebra satisfying the following relations:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\delta_{m+n,0}c$$

and is a Virasoro Lie algebra (a central extension of the Witt Lie algebra of vector fields: $\langle t^n \frac{d}{dt} \rangle$), with $L(n) = -t^{n+1}\frac{d}{dt}$ on $\mathbb{C}[t, t^{-1}]$

- $L(-1) = D$.

On a VA, there could be many conformal structures!

Example of a vertex algebra Any commutative algebra A is a vertex algebra with

$Y(a, z) = a_{-1} : V \to V$ is the multiplication of a on A

and D and $\mathbf{1}$ the identity of A. It is also a vertex operator algebra with trivial Virasoro Lie algebra module structure, i.e., $\omega = 0$.

V**-modules:** (M, Y_M) (for VA).

$$Y_M(v, z) = \sum v_n z^{-n-1}, \quad v_n \in \mathrm{End}_{\mathbb{C}}(M)$$

which are fields on M and the locality property holds and the associativity holds:

$$(z_0 + z_1)^k (Y_M(Y(u, z_0)v, z_1)x$$
$$= (z_0 + z_1)^k Y_M(u, z_0 + z_1)Y_M(v, z_1))x$$

for all $u, v \in V$ and $x \in M$ and some $k = k(u, x) > 0$.

If A is a commutative algebra and viewed as vertex algebra, then vertex algebra modules are exactly the modules of the commutative algebra.

Fact: the module category for a vertex algebra is an abelian category.

If V is a vertex operator algebra, any vertex algebra module (M, Y_M^V) automatically has a module structure of the Virasoro Lie algebra defined by the operators $L_M^V(n)$ on M from

$$Y_M^V(\omega^V, z) = \sum_n L_M^V(n) z^{-n-2}.$$

There are more conditions on representations of VOA: $L(0)$ is semisimple with finite dimensional eigenspaces and eigenvalues (weights) should be bounded below (similar to category $\mathcal{O}$ but corresponding to lowest weights).

3. Semi-conformal subalgebras

A VA-homomorphism

$f : (W, Y^W, \mathbf{1}^W) \to (V, Y^V, \mathbf{1}^V)$

$$f(Y^W(w_1, z)w_2) = Y^V(f(w_1), z)f(w_2),\ f(\mathbf{1}^W) = \mathbf{1}^V.$$

For a VA-homomorphism $f : (W, Y^W, \mathbf{1}^W, \omega^W) \to (V, Y^V, \mathbf{1}^V, \omega^V)$.

f is conformal if $f(\omega^W) = \omega^V$, which is equivalent to

$$f \circ L^W(n) = L^V(n) \circ f \text{ for all } n \in \mathbb{Z}$$

i.e., a homomorphism of Virasoro modules.

f is semi-conformal if

$$f \circ L^W(n) = L^V(n) \circ f \text{ for all } n \geq -1.$$

If $f : W \subseteq V$, then $Y^W = Y^V|_W$ and we call $(W, Y^W, \mathbf{1}^W, \omega^W$ a conformal (semi-conformal) subVOA of $(V, Y^V, \mathbf{1}^V, \omega^V)$.

Definition 1. For any VOA $(V, Y, \mathbf{1}, \omega)$ we define

- $\mathrm{ScAlg}(V, \omega^V) = \{(W, \omega^W) \subseteq (V, \omega^V)$ semi conf. subalg$\}$
- $\mathrm{Sc}(V, \omega^V) = \{\omega' \in V |$ a semi-conformal vector$\}$

Theorem 1. *For any vertex operator algebra $(V, Y, \mathbf{1}, \omega)$, the set $\mathrm{Sc}(V, \omega^V)$ of semi-conformal vectors of (V, ω^V) is an affine algebraic variety.*

In fact, the equations for the variety $\mathrm{Sc}(V, \omega^V)$ are

$$\begin{cases} L(0)\omega' = 2\omega'; \\ L(1)\omega' = 0; \\ L(2)\omega' = \frac{1}{2}c\mathbf{1}; \\ L'(-1)\omega' = L(-1)\omega'; \\ L(n)\omega' = 0, n \geq 3. \end{cases} \tag{1}$$

Theorem 2. *For any vertex operator algebra* $(V, Y, \mathbf{1}, \omega)$ *and any vertex subalgebra* W*, there is at most one conformal structure* $\omega^W \in W$ *on* W *such that* (W, ω^W) *is semi-conformal vertex operator subalgebra.*

Theorem 3. *If* $(W, Y^W, \mathbf{1}^W, \omega^W) \subseteq (V, Y^V, \mathbf{1}^V, \omega^V)$ *is a semi-conformal subVOA, then* $\mathrm{Sc}(W, \omega^W) \subseteq \mathrm{Sc}(V, \omega^V)$

Affine vertex algebras

Example 1. • Let $\mathfrak{g}$ be a Lie algebra, with a non-degenerate invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$. Invariant means $\langle [x, y], z \rangle = \langle x, [y, z] \rangle$.

• The corresponding affine Lie algebra with C central is

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}C$$

with Lie structure

$$[xt^n, yt^m] = [x,y]t^{n+m} + n\delta_{n+m,0}C.$$

$\widehat{\mathfrak{g}}_+ = \mathfrak{g}[t] \oplus \mathbb{C}C \subseteq \widehat{\mathfrak{g}}$ is a Lie subalgebra.

$V_{\widehat{\mathfrak{g}}}(l,0) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{g}}_+)} \mathbb{C}_l$ (the Verma module) has a vertex algebra structure.

- $C = l \in \mathbb{C}$ is called the level.
- $v^+ = 1 \otimes 1$ is the generator of the $\widehat{\mathfrak{g}}$-module, i.e, the highest weight vector.
- $L_{\widehat{\mathfrak{g}}}(l,0)$ the irreducible quotient of $V_{\widehat{\mathfrak{g}}}(l,0)$ as $\widehat{\mathfrak{g}}$-module.
- Both $V_{\widehat{\mathfrak{g}}}(l,0)$ and $L_{\widehat{\mathfrak{g}}}(l,0)$ have a vertex algebra structure such that

$$Y(xt^{-1}v^+, z) = \sum_{n\in\mathbb{Z}} xt^n z^{-n-1}$$

with xt^n acting on $\hat{g}$-modules. With a few exceptions of $l \in \mathbb{C}$.

- Both $V_{\widehat{\mathfrak{g}}}(l,0)$ and $L_{\widehat{\mathfrak{g}}}(l,0)$ have a conformal structure making them as vertex operator algebras.

- Certain irre. $\widehat{\mathfrak{g}}$-modules $L_{\widehat{\mathfrak{g}}}(l,\lambda)$ are irre. modules for the both VOAs.

Here λ can be thought as irreducible $\mathfrak{g}$-modules.

Example 2. • $\mathfrak{h} \subseteq \mathfrak{g}$ is a subalgebra such that the restriction of the bilinear form $\langle \cdot, \cdot \rangle$ is degenerate and $\widehat{\mathfrak{h}}$ is a subalgebra of $\widehat{\mathfrak{g}}$.

• $L_{\widehat{\mathfrak{h}}}(l,0) = U(\widehat{\mathfrak{h}})v^+ \subseteq L_{\widehat{\mathfrak{g}}}(l,0)$ is an irr. $\widehat{\mathfrak{h}}$-module and has VOA structure. It is not a subVOA, but a semi-comformal subVOA of $L_{\widehat{\mathfrak{g}}}(l,0)$.

• If $\mathfrak{h}$ is a maximal torus, $L_{\widehat{\mathfrak{h}}}(l,0) = V_{\widehat{\mathfrak{h}}}(l,0)$ is the Heisenberg VOA.

4. Centralizers in VOA

For any vertex algebra $(V, Y^V, \mathbf{1}^V)$ and any subset S of V, the *centralizer*

$$C_V(S) = \{v \in V \mid [Y^V(v,z_1), Y^V(s,z_2)] = 0, \forall, s \in S\}.$$

Consequences:

• $C_V(S)$ is always a vertex subalgebra.

• $C_V(S) = C_V(\langle S \rangle)$ with $\langle S \rangle$ being the vertex subalgebra generated by S.

• $C_W(V) = \{w \in W \mid w_n(v) = 0 \; \forall n \geq 0, \forall v \in V\}$

• $C_W(V) = \{w \in W \mid v_n(w) = 0 \; \forall n \geq 0, \forall v \in V\}$

• $C_W(V)$ is a sub VA of W.

• $C_V(V)$ is called the center of the VA V (always a commutative associative algebra).

- $C_V(W) = \text{hom}_{W\text{-Mod}}(W, V)$, space of all W-module homorphisms of $W \subseteq V$ is a vertex subalgebra.

If $(W, Y^W, \mathbf{1}^W, \omega^W) \subseteq (V, Y^V, \mathbf{1}^V, \omega^V)$ are VOAs, $C_V(W)$ needs not be a VOA.

Theorem **4.** *If $(W, Y^W, \mathbf{1}^W, \omega^W) \subseteq (V, Y^V, \mathbf{1}^V, \omega^V)$ is a semi-conformal subVOA, then $C_V(W)$ also a semi-conformal sub VOA with $\omega^{C_V(W)} = \omega^V - \omega^W$.*

- $C_V(W) = \ker(L^W(-1) : V \to V)$

where $Y^V(\omega^W, z) = \sum L^W(n) z^{-n-2}$

- $C_V(V) = \mathbb{C}\mathbf{1}^V$ if V is a simple VOA.

A vertex algebra is called *central* if $C_V(V) = \mathbb{C}\mathbf{1}$.

Theorem **5.** *If $(W, Y^W, \mathbf{1}^W, \omega^W) \subseteq (V, Y^V, \mathbf{1}^V, \omega^V)$ is a semi-conformal subVOA, then the map*

$$\text{Sc}(W, \omega^W) \times \text{Sc}(C_V(W), \omega^{C_V(W)}) \to \text{Sc}(V, \omega^V)$$

defined by $(\omega', \omega'') \mapsto \omega' + \omega''$ is injective.

Poset structure on $\text{Sc}(V, \omega)$

For each $\omega' \in \mathrm{Sc}(V, \omega)$,

$$V(\omega') = C_V(\omega - \omega')$$

The map $\mathrm{ScAlg}(V, \omega) \to \mathrm{Sc}(V, \omega)$
$\omega' \mapsto V(\omega')$ is an injection. is a semi-conformal subalgebra of V.

Definition 2. We say $\omega' \leq \omega''$ if $V(\omega') \subseteq V(\omega'')$.

There is an order reversing map $\mathrm{Sc}(V, \omega) \to \mathrm{Sc}(V, \omega)$ such that $\omega' \mapsto \omega - \omega'$.

Example 3. For for simple $\mathfrak{g}$ and $\mathfrak{h} \subseteq \mathfrak{g}$ Cartan subalgebra $L_{\widehat{\mathfrak{h}}}(l, 0) \subseteq L_{\widehat{\mathfrak{g}}}(l, 0)$. The semiconformal sub VOA $K(\mathfrak{g}, l) := C_{L_{\widehat{\mathfrak{g}}}(l,0)}(L_{\widehat{\mathfrak{h}}}(l, 0))$ is called a parafermion studied intensively by physists.

Conjecture 1. $K(\mathfrak{g}, l)$ *is always rational!*

More general case is speculated. If W is rational and $V \subseteq W$ is semi-conformal and rational, then $C_W(V)$ is also rational.

5. Tensor Products

For two VAs V' and V'', the tensor product VA structure on $V' \otimes V''$ is defined by

$$Y^{V'\otimes V''}(v' \otimes v'', z) = Y^{V'}(v', z) \otimes Y^{V''}(v'', z)$$

and $\mathbf{1}_{V'\otimes V''} = \mathbf{1}_{V'} \otimes \mathbf{1}_{V''}$.

We set $W = V' \otimes V''$ and $V = V' \otimes \mathbf{1}^{V"}$. $C_W(V) \supseteq \mathbf{1}^{V'} \otimes V''$. If both V' and V'' are VOAs with $\omega^{V'}$ and $\omega^{V''}$, then $V' \otimes V''$ is also a VOA with

$$\omega^{V'\otimes V''} = \omega' \otimes \mathbf{1}^{V''} + \mathbf{1}^{V'} \otimes \omega^{V''}.$$

Thus $V' \otimes \mathbf{1}''$ is a semi-conformal subalgebra of $V' \otimes V''$ and $C_{V'\otimes V''}(V' \otimes \mathbf{1}^{V''})$ also a semi-conformal in $V' \otimes V''$ with conformal element $\mathbf{1}^{V'} \otimes \omega^{V"}$.

Proposition 1. $C_{V'\otimes V''}(V' \otimes \mathbf{1}^{V''}) = C_{V'}(V') \otimes V''$. *In particular, If V' is a simple vertex operator algebra, then $C_{V'\otimes V''}(V' \otimes \mathbf{1}^{V''}) = \mathbf{1}^{V'} \otimes V''$.*

Proposition 2. *If V' and V'' are two simple VOAs, then $V' \otimes V''$ is a simple VOA.*

Example 4. For a finite dim. simple Lie algebra $\mathfrak{g}$, $L_{\widehat{\mathfrak{g}}}(l, 0)^{\otimes n}$ is a simple VOA and $L_{\widehat{\mathfrak{g}}}(nl, 0) \subseteq L_{\widehat{\mathfrak{g}}}(l, 0)^{\otimes n}$ is a semiconformal sub VOA.

Example 5. If L is an even lattice and V_L is a lattice

VOA, then $V_L^{\otimes n} \cong V_{L^{\times n}}$. And $V_{\sqrt{n}L} \subseteq V_{L^{\times n}}$ is a semi-conformal subVOA.

Question 1. *Decompose* $L_{\widehat{\mathfrak{g}}}(l,0)^{\otimes n}$ *as* $L_{\widehat{\mathfrak{g}}}(nl,0)$*-modules.*

More generally, given a composition $(l_1, \cdots l_s)$, and simple $L_{\widehat{\mathfrak{g}}}(l_i,0)$-modules M_i, then $M_1 \otimes \cdots \otimes M_s$ is a module for $L_{\widehat{\mathfrak{g}}}(l_1,0) \otimes \cdots \otimes L_{\widehat{\mathfrak{g}}}(l_s,0)$. $L_{\widehat{\mathfrak{g}}}(l_1 + \cdots + l_s, 0) \subseteq L_{\widehat{\mathfrak{g}}}(l_1,0) \otimes \cdots \otimes L_{\widehat{\mathfrak{g}}}(l_s,0)$ is semi-conformal subVOA.

Question 2. *Then decompose* $M_1 \otimes \cdots \otimes M_s$ *as* $L_{\widehat{\mathfrak{g}}}(l_1 + \cdots + l_s, 0)$*-modules.*

These are Schur-Weyl duality of questions.

6. Heisenberg vertex operator algebras

- Let $\mathfrak{h}$ be a d-dim. vector space (abelian Lie alg.) $\langle \cdot, \cdot \rangle$ a nondegenerate symmetric bilinear form on $\mathfrak{h}$
- $\widehat{\mathfrak{h}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h} \oplus \mathbb{C}C$ is the affiniziation of the abelian Lie algebra $\mathfrak{h}$ with

$$[\beta_1 \otimes t^m, \beta_2 \otimes t^n] = m\langle \beta_1, \beta_2 \rangle \delta_{m,-n} C.$$

- $\widehat{\mathfrak{h}}_+ = \mathbb{C}[t] \otimes \mathfrak{h} \oplus \mathbb{C}C$ is an Abelian subalgebra.
- For $\forall \lambda \in \mathfrak{h}$, we can define an one-dimensional $\widehat{\mathfrak{h}}^{\geq 0}$-module $\mathbb{C}e^{\lambda}$ by the actions $(h \otimes t^m) \cdot e^{\lambda} = \langle \lambda, h \rangle \delta_{m,0} e^{\lambda}$

and $C \cdot e^{\lambda} = e^{\lambda}$ for $h \in \mathfrak{h}$ and $m \geq 0$.

- Set

$$V_{\widehat{\mathfrak{h}}}(1, \lambda) = U(\widehat{\mathfrak{h}}) \otimes_{U(\widehat{\mathfrak{h}}_{\geq 0})} \mathbb{C}e^{\lambda} \cong S(t^{-1}\mathbb{C}[t^{-1}] \otimes \mathfrak{h})$$

- Choose an orthonormal basis $\{h_1, \cdots, h_d\}$ of $\mathfrak{h}$

Define $\omega = \frac{1}{2}\sum_{i=1}^{d} h_i(-1)^2 \cdot \mathbf{1} \in V_{\widehat{\mathfrak{h}}}(1, 0)$.

Then $(V_{\widehat{\mathfrak{h}}}(1, 0)$, Y, $\mathbf{1}$, $\omega)$ has a vertex operator algebra structure and

- $(V_{\widehat{\mathfrak{h}}}(1, \lambda), Y)$ becomes an irreducible module of $(V_{\widehat{\mathfrak{h}}}(1, 0)$ for any $\lambda \in \mathfrak{h}$.

- Each $\omega' \in \mathrm{Sc}(V, \omega)$ correspond to a linear map $A_{\omega'}$: $\mathfrak{h} \to \mathfrak{h}$ which is a projection to a regular subspace of $\mathfrak{h}$.

Theorem 6. *1) The map $\rho : \omega' \mapsto \mathrm{Im}(\mathcal{A}_{\omega'})$ is an ordering preserving $\mathrm{Aut}(V_{\widehat{\mathfrak{h}}}(1, 0), \omega)$-equivariant bijection form $\mathrm{Sc}(V_{\widehat{\mathfrak{h}}}(1, 0), \omega)$ to $\mathrm{Reg}(\mathfrak{h})$;*

2) $\mathrm{Sc}(V_{\widehat{\mathfrak{h}}}(1, 0), \omega)$ has exactly $d + 1$ orbits under the group $\mathrm{Aut}(V_{\widehat{\mathfrak{h}}}(1, 0), \omega)$-action and each $0 \leq i \leq d$ corresponds to the orbit

$$\mathrm{Sc}(V_{\widehat{\mathfrak{h}}}(1, 0), \omega)_i = \{\mathfrak{h}' \subset \mathfrak{h} | \mathfrak{h}' \text{ is } i\text{-dim. reg. subsp. of } \mathfrak{h}\}$$

3) *There exists a longest chain in* $\mathrm{Sc}(V_{\widehat{\mathfrak{h}}}(1,0),\omega)$ *such that the length of this chain equals to* d*: there exist* $\omega^1,\cdots,\omega^{d-1}\in \mathrm{Sc}(V_{\widehat{\mathfrak{h}}}(1,0),\omega)$ *such that*

$$0=\omega^0<\omega^1<\cdots<\omega^{d-1}<\omega^d=\omega.$$

Theorem 7. *For each* $\omega'\in \mathrm{Sc}(V_{\widehat{\mathfrak{h}}}(1,0),\omega)$*, the following assertions hold.*

1) $\mathrm{Im}\,\mathcal{A}_{\omega'}$ *generates a Heisenberg vertex operator algebra*

$$V_{\widehat{\mathrm{Im}\,\mathcal{A}_{\omega'}}}(1,0)=C_{V_{\widehat{\mathfrak{h}}}(1,0)}(<\omega-\omega'>)$$

and $\mathrm{Ker}\,\mathcal{A}_{\omega'}$ *generates a Heisenberg vertex operator algebra*

$$V_{\widehat{\mathrm{Ker}\,\mathcal{A}_{\omega'}}}(1,0)=C_{V_{\widehat{\mathfrak{h}}}(1,0)}(<\omega'>);$$

2) $C_{V_{\widehat{\mathfrak{h}}}(1,0)}(V_{\widehat{\mathrm{Ker}\,\mathcal{A}_{\omega'}}}(1,0))=V_{\widehat{\mathrm{Im}\,\mathcal{A}_{\omega'}}}(1,0)$
$C_{V_{\widehat{\mathfrak{h}}}(1,0)}(V_{\widehat{\mathrm{Im}\,\mathcal{A}_{\omega'}}}(1,0)))=V_{\widehat{\mathrm{Ker}\,\mathcal{A}_{\omega'}}}(1,0)$;

3) $V_{\widehat{\mathfrak{h}}}(1,0)\cong C_{V_{\widehat{\mathfrak{h}}}(1,0)}(<\omega'>)\otimes C_{V_{\widehat{\mathfrak{h}}}(1,0)}(C_{V_{\widehat{\mathfrak{h}}}(1,0)}(<\omega'>))$.

7. Isomorphism Problem

Theorem 8. *Let* (V,ω) *be a nondegenerate simple CFT type vertex operator algebra generated by* V_1*. Assume that* $L(1)V_1 = 0$*. If for each* $\omega' \in \mathrm{Sc}(V,\omega)$*, there are*

$$V \cong C_V(C_V(<\omega'>)) \otimes C_V(<\omega'>) \qquad (2)$$

then (V,ω) *is isomorphic to the Heisenberg vertex operator algebra* $(V_{\widehat{\mathfrak{h}}}(1,0),\omega)$ *with* $\mathfrak{h} = V_1$*.*

Theorem 9. *Let* (V,ω) *be a nondegenerate simple CFT type vertex operator algebra generated by* V_1*. Assume* $\dim V_1 = d$ *and* $L(1)V_1 = 0$*. If there exists a chain* $0 = \omega^0 < \omega^1 < \cdots < \omega^{d-1} < \omega^d = \omega$ *in* $\mathrm{Sc}(V,\omega)$ *such that* $\dim C_V(C_V(<\omega^i - \omega^{i-1}>))_1 \neq 0,\ for\ i = 1,\cdots,d$*, then* V *is isomorphic to the Heisenberg vertex operator algebra* $(V_{\widehat{\mathfrak{h}}}(1,0),\omega)$ *with* $\mathfrak{h} = V_1$*.*

THANK YOU!