GIT-Equivalence and Semi-Stable Subcategories of Quiver Representations

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November 21, 2016

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- rep(Q) is the category of finite dimensional quiver representations.

Euler Inner Product and Semi-stability

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Let $\alpha \in \mathbb{Q}^{Q_0}$.
A representation $V \in \text{rep}(Q)$ is said to be $\langle \alpha, - \rangle$-semi-stable if:

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for all subrepresentations $V' \leq V$.

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Likewise, it is $\langle \alpha, - \rangle$-stable if the inequality is strict for proper, non-trivial subrepresentations.

Schur Representations and Generic Dimension Vectors

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We say that $\beta' \hookrightarrow \beta$ if every β-dimensional representation has a subrepresentation of dimension β'.

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And respectively, β is $\langle\alpha,-\rangle$-stable if the second inequality is strict for $\beta' \neq 0, \beta$.

The cone of effective weights

The cone of effective weights for a dimension vector β:

$$\mathcal{D}(\beta) = \{\alpha \in \mathbb{Q}^{Q_0} | \langle \alpha, \beta \rangle = 0, \langle \alpha, \beta' \rangle \leq 0, \beta' \hookrightarrow \beta\}$$

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Theorem (Schofield)

β is a Schur root if and only if
$\mathcal{D}(\beta)^\circ = \{\alpha \in \mathbb{Q}^{Q_0} | \langle \alpha, \beta \rangle = 0, \langle \alpha, \beta' \rangle < 0 \ \forall \ \beta' \hookrightarrow \beta, \beta \neq 0, \beta\}$ is non-empty if and only if β is $\langle \beta, - \rangle - \langle -, \beta \rangle$-stable.

Big Question

Two rational vectors $\alpha_1, \alpha_2 \in \mathbb{Q}^{Q_0}$, are said to be **GIT-equivalent** (or **ss-equivalent**) if:

$$\mathrm{rep}(Q)^{ss}_{\langle \alpha_1, - \rangle} = \mathrm{rep}(Q)^{ss}_{\langle \alpha_2, - \rangle}$$

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Colin Ingalls, Charles Paquette, and Hugh Thomas gave a characterization in the case that Q is tame, which was published in 2015. Their work was motivated by studying what subcategories of rep(Q) arise as semi-stable-subcategories, with an eye towards forming a lattice of subcategories.

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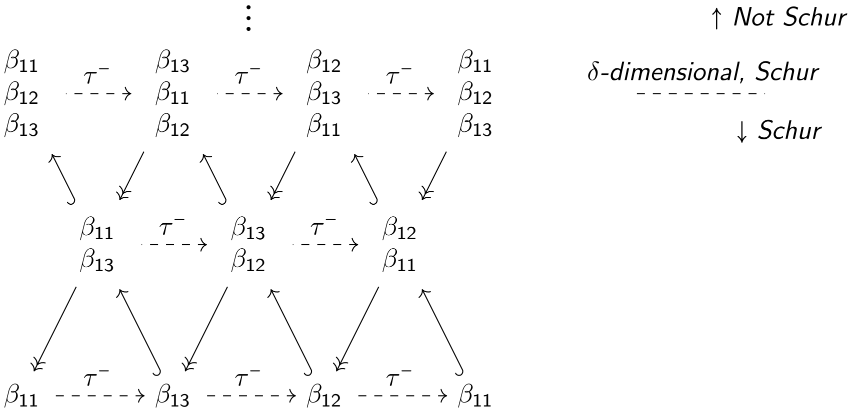
- Similarly for injectives/preinjectives
- Remaining indecomposables occur in tubes
    - Homogeneous tubes (infinitely many)
    - Finitely many non-homogeneous tubes

Non-Homogeneous tubes

Now for example, a rank 3 tube looks like:



Previous work on the tame case

IPT did the following:

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- Set I to be the multi-index $(a_1, \ldots, a_N)$, where $1 \leq a_i \leq r_i$, and R to be the set of all permissible such multi-indices.
- Define the cone C_I to be the rational convex polyhedral cone generated by δ, together with $\beta_{i,j}$, **except for** β_{ia_i}.

Previous work on the tame case

Define $\mathcal{J} = \{C_I\}_{I \in R} \cup \{\mathcal{D}(\beta)\}_{\beta}$, where β is a real Schur root. Set $\mathcal{J}_\alpha = \{\mathcal{C} \in \mathcal{J} | \alpha \in \mathcal{C}\}$.

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Theorem (Ingalls, Paquette, Thomas)

For $\alpha_1, \alpha_2 \in \mathbb{Z}^{Q_0}$, we have that α_1 and α_2 are GIT equivalent if and only if $\mathcal{J}_{\alpha_1} = \mathcal{J}_{\alpha_2}$.

The GIT-cone

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The GIT-fan associated to (Q,β) is:

$$\mathcal{F}(\beta) = \{\mathcal{C}(\beta)_\alpha \mid \alpha \in \mathcal{D}(\beta)\} \cup \{0\}$$

A few remarks about fans

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(Keicher, 2012) Let $\Sigma \subseteq \mathbb{Q}^n$ be a pure n-dimensional fan with convex support $|\Sigma|$, and let $\tau \in \Sigma$ be such that $\tau \cap |\Sigma|^\circ \neq \varnothing$. Then τ is the intersection over all $\sigma \in \Sigma^{(m)}$ satisfying $\tau \leq \sigma$.

Main Theorem

Set

$$\mathcal{I} = \{C(\beta)_\alpha | \beta \text{ is a Schur root and } C(\beta)_\alpha \text{ is maximal}\}$$

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Theorem (Theorem 1)

Let Q be a connected, acyclic quiver. For $\alpha_1, \alpha_2 \in \mathbb{Q}^{Q_0}$, $\alpha_1 \sim_{GIT} \alpha_2$ if and only if $\mathcal{I}_{\alpha_1} = \mathcal{I}_{\alpha_2}$

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That is, we have a collection of cones parametrized by Schur roots which characterizes GIT-equivalence classes.

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Now, if β is arbitrary, use a JH-filtration to break it into a sum of $\langle \alpha_1, - \rangle$-stable factors.

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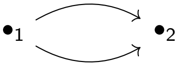
- Invoke a result that $\mathcal{C}(\beta)_{\alpha} = \bigcap \Omega(W)$ (Chindris, "On GIT Fans for Quivers")

Example

Let $Q = \tilde{\mathbb{A}}_1$:



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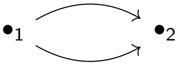

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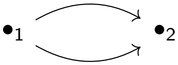

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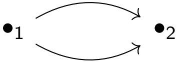

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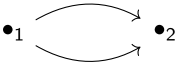

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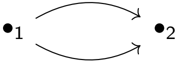

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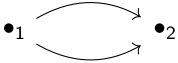
Example

Let $Q = \tilde{\mathbb{A}}_1$:



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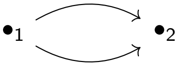

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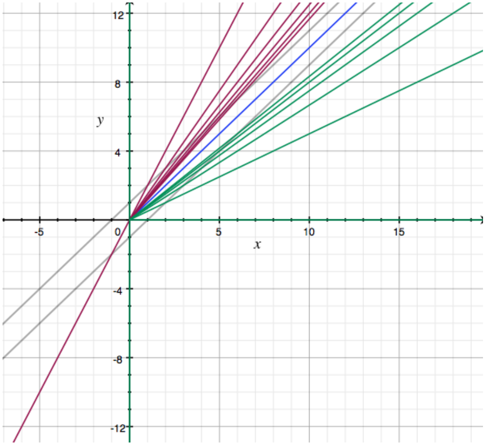
Lastly, $\mathcal{D}(\delta)$ is generated by δ.

Example



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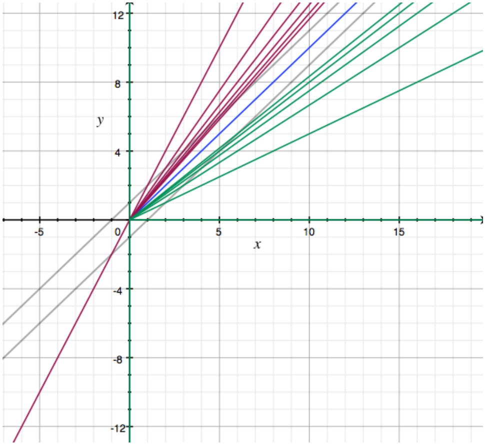
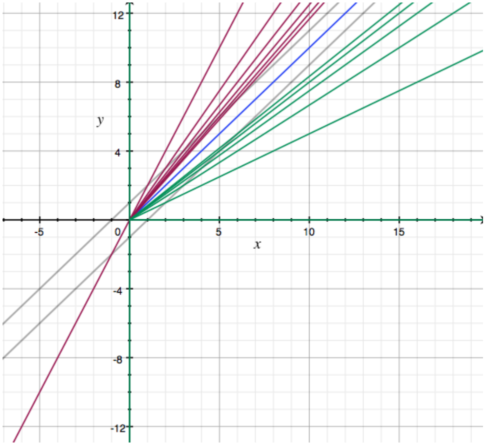


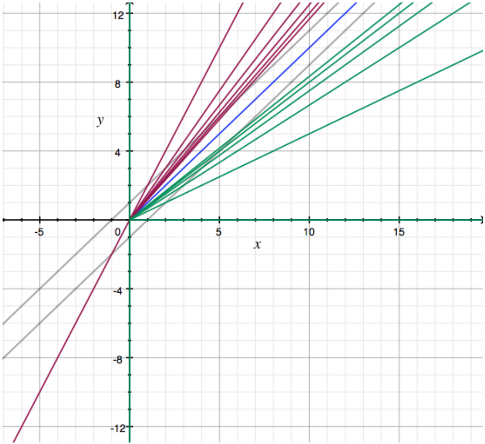
Rays extending through lattice points of $y = x + 1$ and $y = x - 1$

Example



Two weights α_1 and α_2 are GIT equivalent if and only if they are on the same collection of rays.

Example



In this case, since the intersection of any two rays is $(0,0)$, we have that α_1, α_2 are GIT-equivalent if they are

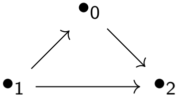
- both $= (0,0)$
- both in the same ray, i.e., $\alpha_1 = \lambda\alpha_2$ for some $\lambda \in \mathbb{Q}$

Further Questions

- How can we get our hands on these maximal GIT-cones of Schur roots for wild quivers?
- Would a similar result hold, using similar techniques, for quivers with relations?

Example

Let $Q = \tilde{\mathbb{A}}_2$:



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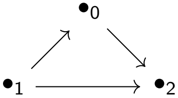

We want to give an idea of the cones in $\mathcal{I}$. Recall that

$$\mathcal{I} = \{\mathcal{C}(\delta)_{\alpha_I}\}_{I \in R} \cup \{\mathcal{D}(\beta)\}_{\beta}$$

where the union is over all real Schur roots β.

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Each one of these real Schur roots will correspond to a $\mathcal{D}(\beta) \in \mathcal{I}$.

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For example, if we take $\beta = (0,0,1)$, we have $\mathcal{D}(\beta)$ is generated by $-\underline{\dim} P_0 = (-1,0,-1)$, $-\underline{\dim} P_1 = (-1,-1,-2)$ and $(1,0,1)$, which is $-\langle -, \beta \rangle$-stable.

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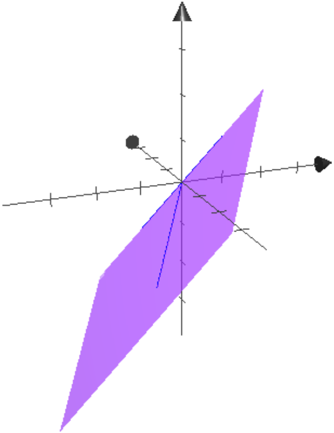
Thus, $\mathcal{D}(\beta)$ looks like:



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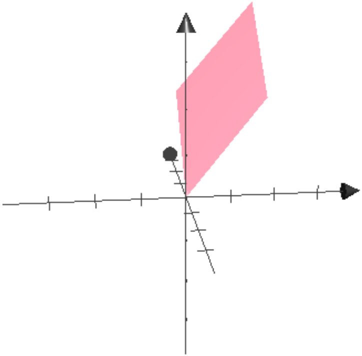

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Now, $\beta_{11} = (0,1,1)$ and $\beta_{12} = (1,0,0)$ are themselves real Schur roots, and so $\mathcal{D}(\beta_{11})$ and $\mathcal{D}(\beta_{12})$ are in $\mathcal{I}$.

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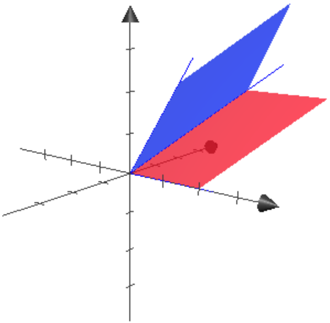

[Animation with many of the cones from $\mathcal{I}$]