Canonical Join Representations for Torsion Classes

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Goal: Torsion Classes

- Study the combinatorics of the lattice $\operatorname{tors}\Lambda$ for a finite-dimensional associative algebra Λ.
- Describe each cover in the lattice via a Schur module
- Describe join-irreducible elements in $\operatorname{tors}\Lambda$.
- Describe the canonical join representation of elements.

Definitions from algebra

Throughout, Λ is a finite-dimensional associative algebra over a field k.

- A *torsion class* over Λ is a class of modules $\mathcal{T}$ closed under extensions and epimorphisms.
- We will focus on the lattice structure of $\operatorname{tors}\Lambda$ with partial order given by inclusions.
- Given any class of modules $\mathcal{S} \subset \operatorname{mod}\Lambda$, $\operatorname{filt}(\mathcal{S})$ denotes the set of modules admiting an $\mathcal{S}$-filtration. I.e., $X \in \operatorname{filt}(\mathcal{S})$ if there is a filtration

$$X = X_n \supsetneq X_{n-1} \ldots \supsetneq X_0 = 0$$

  with $X_i/X_{i-1} \in \mathcal{S}$ for $i = 1, \ldots, n$.
- If $\mathcal{T}$ and $\mathcal{T}'$ are torsion classes, then $\operatorname{filt}(\mathcal{T} \cup \mathcal{T}')$ is the smallest torsion class containing both, the *join*, $\mathcal{T} \vee \mathcal{T}'$.
- Meanwhile, $\mathcal{T} \cap \mathcal{T}' = \mathcal{T} \wedge \mathcal{T}'$ is the meet.

Background

History Born of reflection functors, they can help to understand a module category by relating it to a simpler algebra [formalized by Brenner-Butler]

Happel-Unger Torsion classes can be generated by tilting objects F, and almost-complete tilting modules can be completed it *at most* two ways to a tilting module.

Adachi-Iyama-Reiten Defined the notion of τ-tilting pairs, $(M, P) \in \operatorname{mod}\Lambda \times \operatorname{Proj}\Lambda$, and almost-complete τ-tilting pairs can be completed in *exactly* two ways.

AIR $\operatorname{Fac}(M)$ is a (functorially-finite) torsion class, and torsion classes obtained from an *exchange* adjacent in the poset of torsion classes.

Take-away

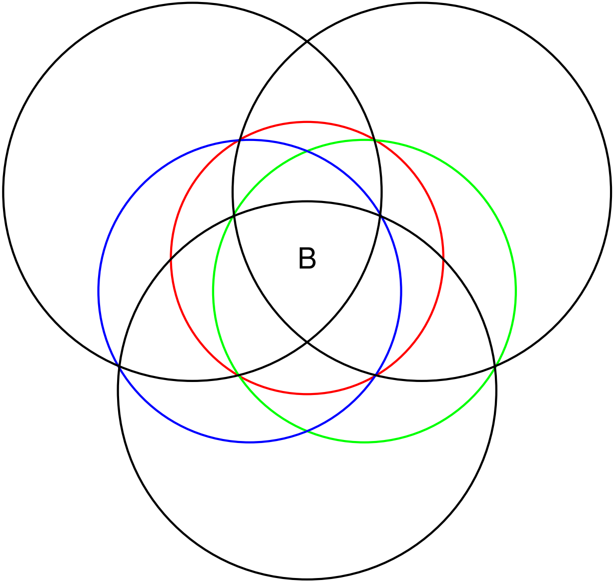
τ-tilting exchange gives information on the poset structure of $\mathrm{f.\,f.\,tors}(\Lambda)$.

A beautiful story

Mizuno ff-tors over preprojective of Dynkin quiver corresponds to weak order on the corresponding Weyl group

Reading For finite Coxeter group, consider the corresponding hyperplane arrangement of reflecting hyperplanes. Hyperplanes *cut* each other into pieces called shards.

Iyama-Reading-Reiten-Thomas (Simultaneous to BCTZ) Study lattice structure of the weak order via representations of preprojective algebra, including interpretation of shards.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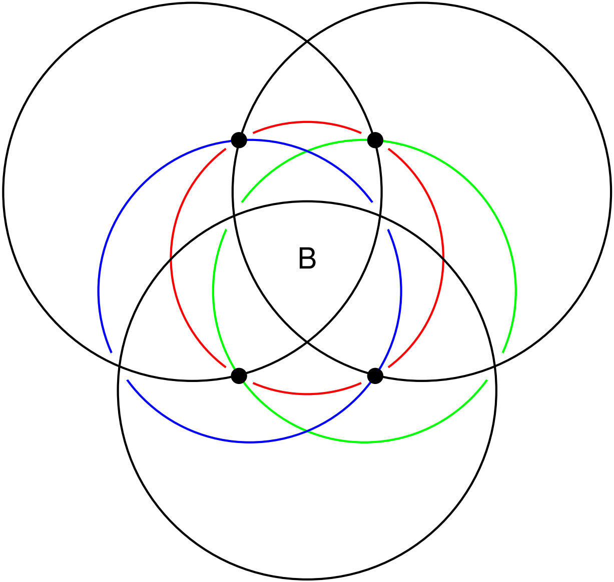
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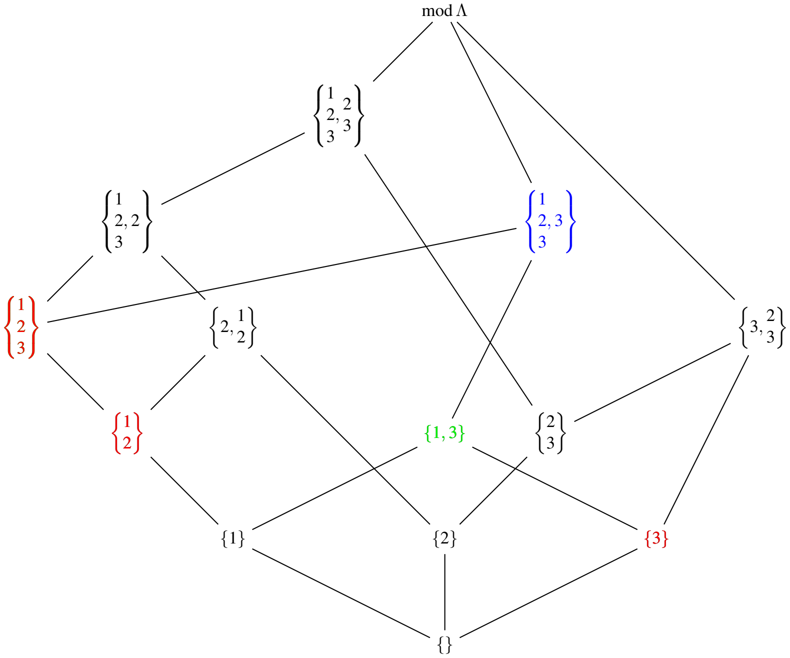
Definitions from lattices

- An element $x \in L$ is *join irreducible* if whenever there is a finite subset $A \subset L$ with $x = \bigvee A$, $x \in A$. (x is *completely join irreducible* if the property holds for arbitrary subsets $A \subset L$.)
- A *join-representation* of $x \in L$ is an expression $x = \bigvee A$ where A is a finite subset of L.
  - A join-representation is *irredundant* if $\bigvee A' < \bigvee A$ for any proper subset $A' \subset A$.
  - A *join-refines* B if for each $a \in A$, there is a $b \in B$ with $a \leq b$.
  - A *canonical join-representation* of x: $x = \bigvee A$, irredundant and minimal with respect to join refinement.

$1233 = 12 \vee 3$



Minimal extending modules

To each cover $\mathcal{T} \lessdot \mathcal{T}'$ in the lattice of torsion classes, we associate an indecomposable Schur module.

Definition

Let $\mathcal{T}$ be a torsion class. An indecomposable M is called *minimal extending module* for $\mathcal{T}$ if

1. All proper factors of M lie in $\mathcal{T}$;
2. For every non-split exact sequence

$$0 \to M \to X \to T \to 0,$$

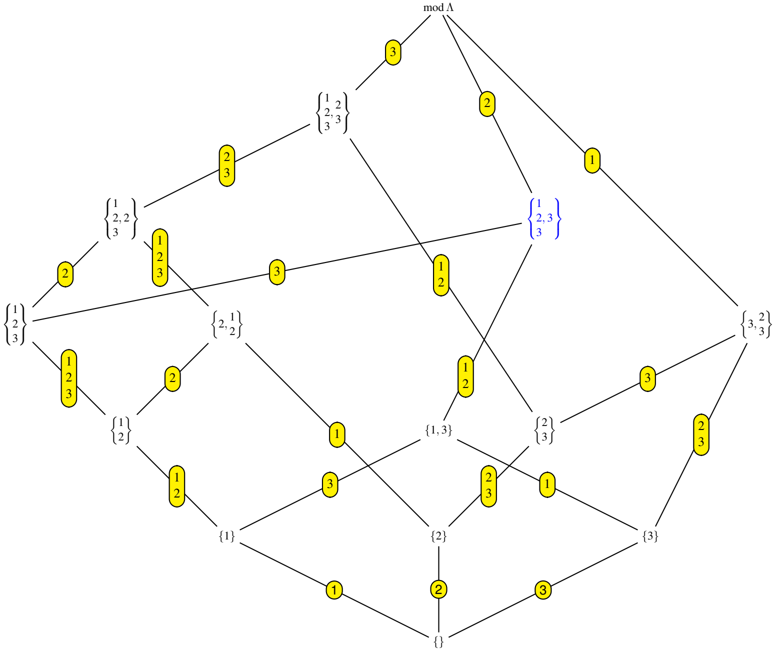
   $T \in \mathcal{T}$ implies $M \in \mathcal{T}$.
3. $\mathrm{Hom}_\Lambda(\mathcal{T}, M) = 0$

Covers are given by minimal extending modules

Theorem (BCTZ)

The torsion class $\mathcal{T}$ admits a cover $\mathcal{T}'$ if and only if there exists a minimal extending module M for $\mathcal{T}$ which lies in $\mathcal{T}'$.

- In case one of the equivalent conditions holds, $\mathcal{T}' = \mathrm{filt}(\mathrm{ind}(\mathcal{T}) \cup \{M\})$.
- M is a Schur module (i.e., $\mathrm{End}_\Lambda(M) = k$, sometimes called a *brick*).


mod Λ
{1, 3}
{1}
{2}
{3}
{}

Each edge of the Hasse diagram can be labeled by a Schur module. But which ones?

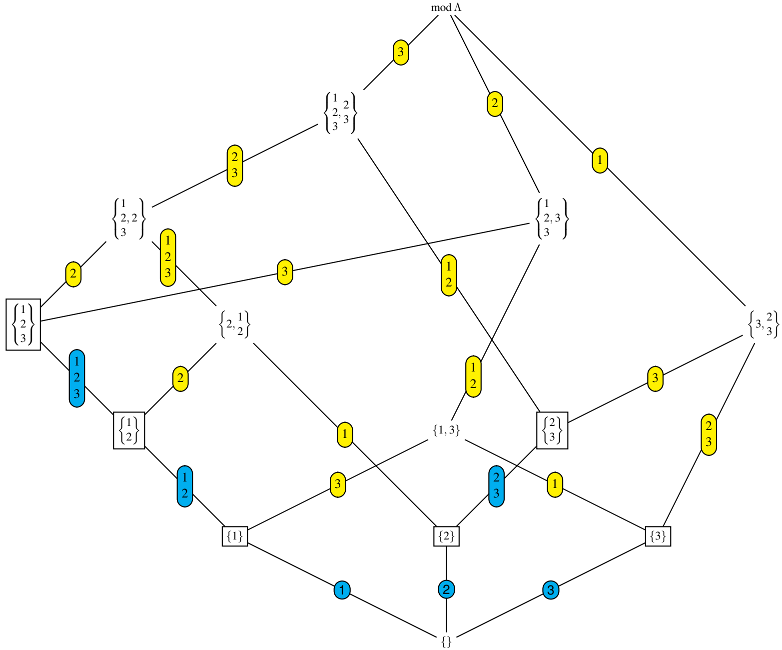
Theorem (BCTZ)

There is a bijection between completely join-irreducible torsion classes over Λ and Schur Λ-modules.

Proof.

1. "Completely join-irreducible" means "has exactly one lower cover"
2. If M is Schur, show $\mathcal{T}_M := \mathrm{filt}(\mathrm{Gen}(M))$ is completely join-irreducible.
3. Restrict ϕ to those covers $\{\mathcal{T} \lessdot \mathcal{T}'\}$ for which $\mathcal{T}'$ is completely join-irreducible.
4. Show that if $\phi(\mathcal{T} \lessdot \mathcal{T}') = M$ then $\mathcal{T}' = \mathcal{T}_M$.

□


mod Λ
{1, 3}
{1}
{2}
{3}
{}

Canonical join-representations

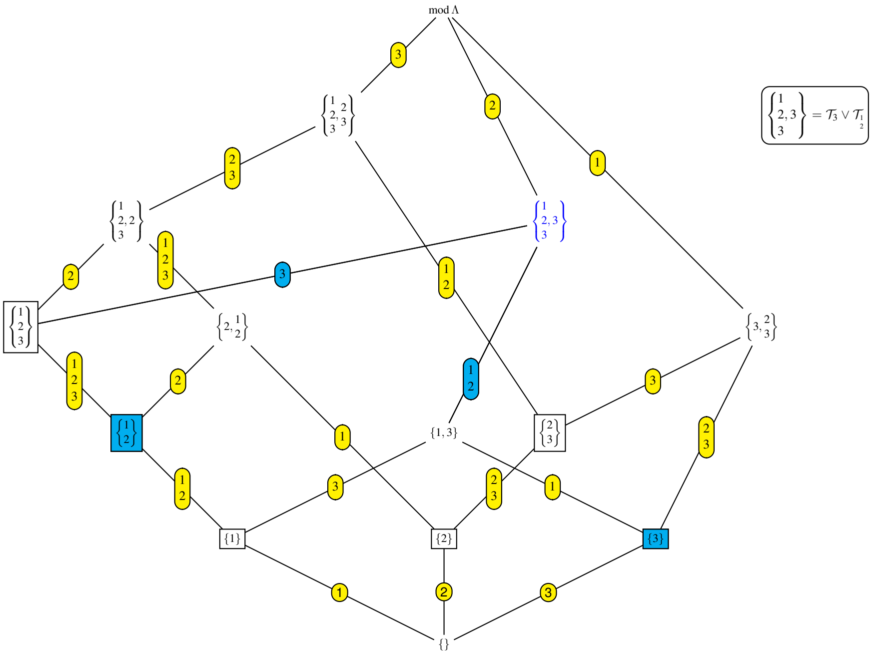
- If $x = \bigvee A$ is the canonical join-representation of x, then each element $a \in A$ is join-irreducible.
- If $\mathcal{T}$ is a torsion class, write $\mathrm{cov}^{\downarrow}(\mathcal{T})$ for the set of torsion classes $\mathcal{S}$ such that $\mathcal{S} \lessdot \mathcal{T}$.
- Call $\mathcal{T}$ *accessible* if for every torsion class $R < \mathcal{T}$, there is an element $\mathcal{S} \in \mathrm{cov}^{\downarrow}(\mathcal{T})$ with $R \leq \mathcal{S}$.
- Let $\mathrm{face}^{\downarrow}(\mathcal{T})$ be the set of Schur modules representing the covers in $\mathrm{cov}^{\downarrow}(\mathcal{T})$.

Theorem (BCTZ)

Suppose that $\mathcal{T}$ is a torsion class. If $\mathcal{T}$ is accessible then

$$\mathcal{T} = \bigvee_{M \in \mathrm{face}^{\downarrow}(\mathcal{T})} \mathcal{T}_M.$$

Is this the CJR?



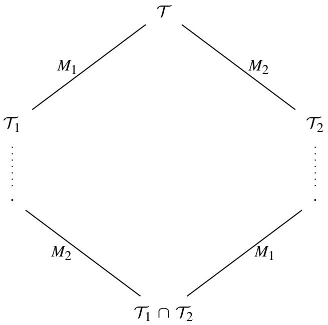
mod Λ
$= \mathcal{T}_3 \vee \mathcal{T}_{\substack{1\\2}}$
{1}
{2}
{3}
{1, 3}
{}

Digging Deeper

Proposition

If $\mathcal{T}$ is a torsion class, and M, N are distinct elements in $\mathrm{face}^{\downarrow}(\mathcal{T})$, then $Hom_{\Lambda}(M,N) = Hom_{\Lambda}(N,M) = 0$.

Proof.



- Can show $M_1 \in \mathcal{T}_2$ and $M_2 \in \mathcal{T}_1$
- (can even show more)
- Property 3 says $\mathrm{Hom}_{\Lambda}(\mathcal{T}_i, M_i) = 0$

□

Hom-configurations

Theorem

Let Λ be a finite dimensional associative algebra of and $M_1, \dots, M_k$ a collection of Schur modules with

$$\dim Hom_\Lambda(M_i, M_j) = 0$$

for all i, j. Then $\bigvee_i \mathcal{T}_{M_i}$ is an accessible torsion class with lower faces M_i.

Corollary (BCTZ)

With the same assumptions as above, $\bigvee_{i \in I} \mathcal{T}_{M_i}$ is a canonical join representation if and only if $\dim_k Hom_\Lambda(M_i, M_j) = 0$ for all $i, j \in I$. In particular, for finite representation type, hom-configurations and torsion classes are in bijection.

Beware: This does not give *all* CJRs, just CJRs with downset explained by their lower faces.

Further work

Lattice quotients (time permitting)

1. Suppose $\Lambda' = \Lambda/I$ for some two-sided ideal I. Then $\mathrm{tors}(\Lambda')$ is a lattice quotient of $\mathrm{tors}(\Lambda)$.
2. There are lattice quotients $\mathrm{tors}(\Lambda) \to \mathcal{L}$ *not* corresponding to algebra quotients.
3. Algebraically, a quotient algebra is the choice of setting isomorphic some indecomposables to a direct sum of a sub and a quotient.
4. On particular, certain Schur modules must be excised.

Thank you!