Nilpotence and generation in the stable module category

Jon F. Carlson
University of Georgia

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Joint work with Dave Benson

Notation.

$k =$ a field of characteristic p, algebraically closed.

$G =$ a finite group.

All kG-modules are finitely generated.

For M a kG-module, let $M^* = \mathrm{Hom}_k(M, k)$, the k-dual.

Recall that $\mathrm{Hom}_k(M, N) \cong M^* \otimes N$

Let $\mathrm{Tr} : M^* \otimes M \to k$ be the trace map. Note that if p does not divide the dimension of M, then Tr is split by the map

$$k \longrightarrow M^* \otimes M \cong \mathrm{Hom}_k(M, M)$$

that sends $1 \in k$ to $\mathrm{Id} \in \mathrm{Hom}_{kG}(M, M)$

Basic results.

Theorem: (Benson-Carlson, 1984) Assume that k is algebraically closed. Suppose that M and N are indecomposable modules and that k is a direct summand of $M \otimes N$. Then

1. $\text{Dim}(M)$ is not divisible by p,
2. $N \cong M^*$,
3. the multiplicity of k as a direct summand of $M \otimes N$ is one, and
4. the trace map $\text{Tr} : M \otimes M^* \to k$ is split.

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Corollary Suppose that M and N are kG-modules such that M is indecomposable and has dimension divisible by p. Then any direct summand of $M \otimes N$ has dimension divisible by p.

The Stable Category.

The stable category $\mathbf{stmod}(kG)$ has

objects: Finitely generated kG-modules

and morphisms (for M and N objects):

$$\underline{\underline{\mathrm{Hom}}}_{kG}(M, N) = \frac{\mathrm{Hom}_{kG}(M, N)}{\mathrm{PHom}_{kG}(M, N)}$$

where PHom means homomorphisms that factor through projectives modules.

This is a *tensor* triangulated category. The triangles correspond to exact sequences. The shift functor is Ω^{-1} where $\Omega^{-1}(M)$ is the cokernel of the injective hull $M \to I_M$.

Remember that projective modules are injective modules and *vice versa*.

We say that a kG-module N is *generated in n steps* from a collection of modules $\{M_\alpha\}$ if there is a triangle

$$N_1 \to N_2 \to N \oplus Z$$

in $\mathbf{stmod}(kG)$, for some Z, where N_1 is generated in $n-1$ steps from $\{M_\alpha\}$ and N_2 is a direct sum of shifts of modules M_α in the collection. To begin the induction, N is generated in one step from the $\{M_\alpha\}$ if it is stably isomorphic to a summand of a direct sum of shifts of the modules M_α.

Note that if G is a p-group then k generates $\mathbf{stmod}(kG)$ in at most $\ell =$ Loewy length of kG steps.

$$(\operatorname{Rad}^i(M)/\operatorname{Rad}^{i+1}(M) \text{ is a sum of copies of } k.)$$

We say that a kG-module N is *tensor generated in n steps* from a collection of modules $\{M_\alpha\}$ if it is generated in n steps by the collection of modules $\{M_\alpha \otimes X\}$ with M_α in the original collection and X arbitrary.

Generation

Lemma: If N is tensor generated in n steps from $\{M_\alpha\}$ then so is every module of the form $N \otimes Y$.

The *tensor M-generation number* is the number of steps it takes to generate k from modules of the form $M \otimes X$ (or infinity if k cannot be generated from such modules).

Given a module M, the modules that can be generated from M form a thick subcategory of $\mathbf{stmod}(kG)$.

Suppose that M is a finitely generated kG-module. Then the ring $\mathrm{Ext}^*_{kG}(M, M)$ is a finitely generated module over the cohomology ring $\mathrm{H}^*(G, k) \cong \mathrm{Ext}^*_{kG}(k, k)$. (Take an extension of k by k, tensor it with M, and get an extension of M by M.)

Let $J(M)$ be the annihilator of $\mathrm{Ext}^*_{kG}(M, M)$ in $\mathrm{H}^*(G, k)$.

Let $V_G(k) = \mathrm{Proj}(\mathrm{H}^*(G, k))$ be the spectrum of homogeneous prime ideals in $\mathrm{H}^*(G, k)$.

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Theorem: (Benson-C-Rickard) A thick subcategory of **stmod**(kG) is determined entirely by the support varieties of its objects.

So, if $V_G(M) = V_G(k)$, then M generates k.

Let M and N be kG-modules. We say that a map $f\colon N \to M$ is *tensor nilpotent* if some tensor power $f^{\otimes n}\colon N^{\otimes n} \to M^{\otimes n}$ is null (factors through a projective module). The *degree of tensor nilpotence* is the smallest such n.

Proposition $f : M \to N$ is tensor nilpotent if and only if its restriction to every elementary abelian p-subgroup is tensor nilpotent.

Let $E = \langle g_1, \dots, g_r \rangle \cong (\mathbb{Z}/p)^r$ be an elementary abelian p-group of rank r. We set $X_i = g_i - 1 \in kE$, for $1 \leq i \leq r$. This allows us to regard kE as a truncated polynomial ring,

$$kE = k[X_1, \dots, X_r]/(X_1^p, \dots, X_r^p).$$

If $0 \neq \alpha = (\lambda_1, \dots, \lambda_r) \in \mathbf{A}^r(k)$, we set

$$X_\alpha = \lambda_1 X_1 + \cdots + \lambda_r X_r \in kE.$$

Identifying kE with the restricted enveloping algebra of a commutative Lie algebra, we get a Hopf algebra structure:

$$\Delta(X_\alpha) = X_\alpha \otimes 1 + 1 \otimes X_\alpha.$$

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Theorem: Endow kE with the Lie theoretic Hopf algebra structure. Then a map $f \colon N \to M$ of finitely generated kE-modules is tensor nilpotent if and only if its restriction to every $k[X_\alpha]/(X_\alpha^p)$ is tensor nilpotent.

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Hopf structure

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And even worse:

Proposition: In general, tensor nilpotent maps can have arbitrarily large degree.

Ghosts

Let M be a kG-module. We say that a map of kG-modules $f\colon X \to Y$ is an *M-ghost* if $f_*\colon \widehat{\operatorname{Ext}}^*_{kG}(M,X) \to \widehat{\operatorname{Ext}}^*_{kG}(M,Y)$ is the zero map.

The map $f\colon X \to Y$ is a tensor-M-ghost if it is an $M \otimes U$-ghost for every kG-module U. This is equivalent to the statement that $\operatorname{Id}_M \otimes f\colon M \otimes X \to M \otimes Y$ is null (factors through a projective).

A tensor-k-ghost is null.

More ghosts

Lemma: Let $\iota : k \to M \otimes M^*$ be the adjoint to the identity map $M \to M$, and complete to a triangle

$$N \xrightarrow{f} k \xrightarrow{\iota} M \otimes M^* \longrightarrow \Omega^{-1}(N)$$

in $\mathbf{stmod}(kG)$. Then $f : N \to k$ is a tensor M-ghost. This map is null if and only if the dimension of M is not divisible by p.

The (tensor) *M-ghost number* is the smallest n such that every composite of n (tensor) M-ghosts is null.

Theorem: Given a kG-module M, form the triangle $N \xrightarrow{f} k \to M \otimes M^*$ in $\mathbf{stmod}(kG)$ as above. Then the following statements are equivalent.

(i) The tensor M-generation number is at most n.

(ii) The map $f^{\otimes n}$ is null.

(iii) The tensor M-ghost number is at most n.

These statements hold for some (finite) $n \geq 1$ if and only if the support variety $V_G(M)$ is equal to $V_G(k)$.

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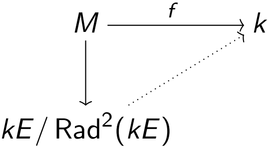
Question: For kG-modules M with $V_G(M) = V_G(k)$, is there a bound on the tensor M-generation number that depends only on G and not on M?

Let $E = \langle g_1, \ldots, g_r \rangle \cong (\mathbb{Z}/p)^r$ be an elementary abelian p-group of rank r, and let $f \colon M \to k$ be a kE-module homomorphism. We say that f is *strongly nilpotent* if, when k is regarded as $kE/\operatorname{Rad}(kE)$, the map f factors through to a map from $kE/\operatorname{Rad}^2(kE)$ to M, as in the following diagram:



Strongly nilpotent

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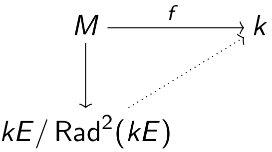

Proposition: Let $f\colon M \to k$ be a map of kE-modules. If f is strongly nilpotent, then f is tensor nilpotent. More precisely, $f^{\otimes r(p-1)}$ is null.

Strong nilpotence does not depend on the Hopf structure.

A bound on generation numbers for modules M with $V_G(M) = V_G(k)$ would follow from the statement that for any such M, the map f in the triangle $N \xrightarrow{f} k \to M \otimes M^*$ is strongly nilpotent.

The statement about generation for an arbitrary G reduces to the same statement for elementary abelian subgroups.

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Theorem: Given a kE-module M, form the triangle $N \xrightarrow{f} k \to M \otimes M^*$ as usual. Then the following are equivalent:
(i) f is strongly nilpotent.
(ii) The trace map
$\mathrm{Tr}_M \colon \mathrm{Ext}^1_{kE}(M, M) \cong \mathrm{Ext}^1_{kE}(k, M^* \otimes M) \to \mathrm{Ext}^1_{kE}(k, k)$ is surjective.