Noncommutative desingularization of orbit closures for some $GL_n$ representations

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Conference on Geometric Methods in Representation Theory 2012
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2 Constructions and formulas for noncommutative desingularization
   - Construction
   - Formula

3 Equivariant quivers

4 Determinantal varieties of symmetric matrices
   - A noncommutative desingularization
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Non-commutative desingularization

Goal: to construct and study noncommutative desingularizations of Orbit closures in representations.
Assume $X$ is singular.

Definition

A non-commutative desingularization of $X$ is a coherent sheaf of associative algebras $\mathcal{A}$, such that

- $\mathcal{A} = \text{End}_X(\mathcal{E})$ for a reflexive coherent sheaf $\mathcal{E}$,
- it has finite homological dimension.

Definition

When $X = \text{Spec } R$ for a Gorenstein ring $R$, a non-commutative desingularization $\mathcal{A}$ is called a non-commutative crepant resolution, if it is maximal Cohen-Macaulay.
Motivation

Tilting bundles

Definition

For a scheme $Z$, a vector bundle $\mathcal{T}il$ is called a tilting bundle over $Z$ if it satisfies the following:

1. it generates $D^b(Z)$, or equivalently $\mathcal{T}il^\perp = 0$ in $D^b(Z)$;
2. $\operatorname{Ext}^i(\mathcal{T}il, \mathcal{T}il) = 0$ for $i > 0$.

If $Z$ is projective and $\{\nabla_\alpha \mid \alpha \in I\}$ is a strong exceptional collection, then $\bigoplus_{\alpha \in I} \nabla_\alpha$ is a tilting bundle.
Geometric tilting theorem

Theorem (Hille, van den Bergh 2007)

For $Z$ a projective scheme over a Noetherian affine scheme of finite type, and $\mathcal{T}il \in D(\text{Qcoh}(Z))$ a tilting object. We have the following

1. $R\text{Hom}_{\mathcal{O}_Z}(\mathcal{T}il, -)$ induces an equivalence

\[ D^b(\text{Coh}(Z)) \cong D^b(\text{End}_{\mathcal{O}_Z}(\mathcal{T}il)-\text{mod}). \]

2. If $Z$ is smooth then $\text{End}_{\mathcal{O}_Z}(\mathcal{T}il)$ has finite global dimension.
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Buchweitz, Leuschke, and van den Bergh’s results

- $H = \text{Hom}_k(E, F)$, where $E$ and $F$ are vector spaces over $k$ of dim. $m$, $n$ resp. ($n \leq m$).
- $H \supseteq \text{Spec } R = \{ \varphi \mid \text{rank } \varphi \leq n - 1 \}$.
- $0 \to R \to F^* \otimes \mathcal{O} \to Q \to 0$, the universal sequence over $\mathbb{P}(F^*)$.
- A desingularization of $\text{Spec } R$, $p' : \mathcal{Z} \to \mathbb{P}(F^*)$: the total space of the vector bundle $E^* \otimes_k Q^*$.

Theorem (Buchweitz, Leuschke, and van den Bergh 2009)

- The vector bundle $p'^* \oplus_{a=0}^n \wedge^a \Omega(a)$ is a tilting bundle over $\mathcal{Z}$.
- Its endomorphism algebra $\Lambda = \text{End}_{\mathcal{O}_\mathcal{Z}}(p'^* \oplus_{a=0}^n \wedge^a \Omega(a))$ is maximal Cohen-Macaulay as an $R$-module, and has finite global dimension.
Buchweitz, Leuschke, and van den Bergh’s results

Theorem (Buchweitz, Leuschke, and van den Bergh 2009)

As a $k$-algebra, $\Lambda$ is isomorphic to the path algebra of the quiver

\[ \begin{array}{cccc}
\alpha_1 & \alpha_2 & \ldots & \alpha_n \\
\beta_1 & \beta_2 & \ldots & \beta_m
\end{array} \]

with relations:

\[
\begin{align*}
\alpha_i \alpha_j + \alpha_j \alpha_i &= 0 = \alpha_i^2 \\
\beta_i \beta_j + \beta_j \beta_i &= 0 = \beta_i^2 \\
\alpha_k (\alpha_i \beta_j + \beta_j \alpha_i) &= (\alpha_i \beta_j + \beta_j \alpha_i) \alpha_k \\
gl(\beta_i \alpha_j + \alpha_j \beta_i) &= (\beta_i \alpha_j + \alpha_j \beta_i) \alpha_1
\end{align*}
\]
Set up

Let $G$ be a reductive group, $P < G$ a parabolic subgroup. Suppose the total space $Z$ of a vector subbundle of $\mathbb{A}^n \times G/P$ over $G/P$ desingularizes an affine subvariety $\text{Spec } R \subseteq \mathbb{A}^n$. 

\[ Z \leftarrow \text{Spec } R \stackrel{i}{\rightarrow} \mathbb{A}^n \]
\[ \text{Spec } R \leftarrow \mathbb{A}^n \rightarrow G/P \]
\[ G/P \times \mathbb{A}^n \rightarrow G/P \]

\[ j \quad q' \quad p' \]

\[ q \quad p \]
Construction

Proposition

Notations as before. Let $\mathcal{T} \text{il}$ be a tilting bundle over $G/P$.

1. If $H^i(Z, p'^* \text{End}_{G/P}(\mathcal{T} \text{il})) = 0 \ \forall i > 0$, then $p'^* \mathcal{T} \text{il}$ is a tilting bundle over $Z$.

2. If moreover, $\text{End}_{Z}(p'^* \mathcal{T} \text{il})$ is maximal Cohen-Macaulay, and $\text{End}_{Z}(p'^* \mathcal{T} \text{il}) \cong \text{End}_{R}(q'^* p'^* \mathcal{T} \text{il})$,

then $\text{End}_{R}(q'^* p'^* \mathcal{T} \text{il})$ gives a noncommutative crepant desingularization of $\text{Spec } R$.

Under the condition of (1), if moreover the exceptional locus of $q' : Z \to \text{Spec } R$ has codimension at least two in both $Z$ and $\text{Spec } R$, and $R$ is an integral domain, then (2) is automatic.
Dual exceptional collection

Let $\mathcal{A}$ be a finite type abelian category. Let $\nabla := \{\nabla_\alpha, \alpha \in I\}$ be a strong exceptional collection in $D^b(\mathcal{A})$.

**Definition**

Another set of objects $\Delta = \{\Delta_\alpha, \alpha \in I\}$ (in bijection with $\nabla$) is called the dual collection to $\nabla$ if

- $\text{Ext}^\bullet(\Delta_\beta, \nabla_\alpha) = 0$ for $\beta > \alpha$,
- $\Delta_\beta \cong \nabla_\beta \mod D_{<\beta}$, where $D_{<\beta}$ is the full triangulated subcategory generated by $\{\nabla_\alpha | \alpha < \beta\}$. 
Exceptional collection for Grassmannians

- $\text{Grass}_{n-r}(E)$, the grassmannian of $(n-r)$-planes in the vector space $E$ over $\mathbb{C}$.
- $0 \rightarrow \mathcal{R} \rightarrow E \times \text{Grass} \rightarrow Q \rightarrow 0$, the tautological exact sequence over Grass.
- An exceptional collection over $\text{Grass}_{n-r}(E)$:
  $\{S_\alpha R^* \mid \alpha \text{ sub-partition of } ((r)^{n-r})\}$, where $S_\lambda$ is the Schur functor corresponding to the partition $\lambda$.
- Its dual exceptional collection: $\{S_{\alpha'} Q[|\alpha|] \mid \alpha \text{ sub-partitions of } (r^{n-r})\}$, where $\alpha'$ is the transpose of $\alpha$, and $|\alpha| = \sum_j \alpha_j$. 
Let $\nabla(G/P) = \{\nabla_\alpha \mid \alpha \in I\}$ be a full exceptional collection consisting of equivariant sheaves over $G/P$ with the dual collection $\Delta$. Assume the resolution $q : Z \to \text{Spec } R$ is $G$-equivariant with $q^{-1}(0) = G/P$, and the only fixed closed point of $\text{Spec } R$ is $\{0\} \subset \text{Spec } R$.

**Theorem (Weyman, Z. 2012)**

Assume moreover that $p^*(\bigoplus_\alpha \nabla_\alpha)$ is a tilting bundle over $Z$, and and its endomorphism algebra $\Lambda \cong \text{End}_R(q_*p^*(\bigoplus_\alpha \nabla_\alpha))$ is a non-commutative desingularization. Then,

1. $S_\alpha = R \text{Hom}(p'^* \text{ til}, u_\ast \Delta_\alpha)$’s are all the equivariant simple objects in $\Lambda\text{-mod}$, where $u : G/P \to Z$ is the zero section;
2. a basis of the vector space $\text{Ext}^1_\Lambda(S_\alpha, S_\beta)^*$ generates $\Lambda$ over $\bigoplus_\alpha \mathbb{C}_\alpha$;
3. with the above generators of $\Lambda$, $\text{Ext}^2_\Lambda(S_\alpha, S_\beta)^*$ generates the relations.
How to calculate Exts

Let $G/P$ be the Grassmannian, and $q' : Z \to \text{Grass}$ be the total space of the vector bundle $S_\delta Q^*$ for some partition $\delta$.

**Proposition**

Assume $p'^* \bigoplus_{\lambda \subset ((n-r)^r)} S_\lambda Q^*$ is a tilting bundle on $Z$. Let $S_\alpha := R \text{Hom}(p'^* \mathcal{T} il, u_* \Delta_\alpha)$’s be the equivariant simples. Then, the Ext’s among them are given by

\[
\text{Ext}^t(S_\alpha, S_\beta) \cong \bigoplus_S \bigoplus_{\lambda \in \wedge^s S_\delta} H^{t-s-|\beta|+|\alpha|}(\text{Grass}, S_\lambda Q^* \otimes S_\beta^* \otimes \otimes S_\alpha^* \mathcal{R}),
\]

where $\wedge^s S_\delta$ stands for the decomposition of $\wedge^s S_\delta \mathbb{C}^{n-r}$ into irreducible representations counting multiplicity.

This can be calculated using Borel-Weil-Bott theorem.
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**Definition**

Let $G$ be a reductive group. An $G$-equivariant quiver is a quiver with an action of $G$ on the vector space spanned by arrows between any two vertices.

The Beilinson quiver

\[
\begin{align*}
\bullet_0 & \rightarrow \alpha_0 \rightarrow \bullet_1 & \rightarrow \alpha_1 \rightarrow & \cdots & \rightarrow \alpha_n \rightarrow & \bullet_n \\
\bullet_0 & \rightarrow \alpha_n \rightarrow & \cdots & \rightarrow \alpha_n \rightarrow & \bullet_n \\
\end{align*}
\]

with relations $\alpha_i \alpha_j - \alpha_j \alpha_i$.

Assume $\dim E = n + 1$ with the natural $GL_{n+1}$ action. The Beilinson quiver can be written as

\[
\begin{align*}
\alpha_0(E) & \rightarrow & \alpha_1(E) & \rightarrow & \alpha_{n-1}(E) \\
\bullet_0 & \rightarrow & \bullet_1 & \rightarrow & \cdots & \rightarrow & \bullet_n \\
\end{align*}
\]

with relations $\alpha_i \alpha_{i+1}(\wedge^2 E)$.

The path algebra $KQ$ has a $G$-action, and the multiplication is equivariant.
Equivariant quiver for Grass$_2(E)$ with $E = \mathbb{C}^4$

with relations:

- $\text{Hom}(\begin{array}{c}
\end{array}, \emptyset): \alpha_1 \alpha_3(\wedge^2 E)$;
- $\text{Hom}(\begin{array}{c}
\end{array}, \emptyset): \alpha_1 \alpha_2(\mathbb{S}_2 E)$;
- $\text{Hom}(\begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}): \alpha_5 \alpha_6(\wedge^2 E)$;
- $\text{Hom}(\begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}): \alpha_4 \alpha_6(\mathbb{S}_2 E)$;
- $\text{Hom}(\begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}): \alpha_2 \alpha_4 - \alpha_3 \alpha_5(\mathbb{S}_2 E \oplus \wedge^2 E)$. 
Representations

We have two notions of bound representations, depending on whether we respect the $G$-action or not. We call them representations and equivariant representations respectively.

**Proposition**

The functor $\Phi := R \operatorname{Hom}_{\mathcal{O}_{\operatorname{Grass}_r(n)}}(\bigoplus_{\alpha \subset (m-r)^r} \mathbb{S}_\alpha \mathcal{R}^*, -)$ induces an equivalence

$$D^b_G(\operatorname{Coh}(X)) \cong D^b_G(\mathbb{C}QK(r, n)/\mathcal{I}\text{-mod}),$$

with quasi-inverse given by $\Psi := - \otimes^L (\bigoplus_{\alpha \subset (m-r)^r} \mathbb{S}_\alpha \mathcal{R}^*).$
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Determinantal varieties of symmetric matrices

- $E$, a vector space
- $\text{Spec } R^s \subseteq \text{Sym}_2 E$, determinantal variety defined by vanishing of minors of rank $r + 1$.
- $\text{Grass}$, the grassmannian of $(n - r)$-planes in $E$
- $0 \to \mathcal{R} \to E \times \text{Grass} \to Q \to 0$, the tautological sequence over $\text{Grass}$.
- $p' : Z^s \to \text{Grass}$, the total space of $\text{Sym}_2 Q^*$ is a commutative desingularization of $\text{Spec } R^s$.
- $\mathcal{I}l_K = \bigoplus_{\alpha \subset ((n-r)r)} S_\alpha Q^*$, the Kapranov's tilting bundle over $\text{Grass}$.

\[
\begin{align*}
Z^s & \xrightarrow{j} Y^s & \xrightarrow{p} \text{Grass} = \text{Grass}_{n-r}(E) \\
& \xrightarrow{q'} \text{Spec } R^s & \xrightarrow{i} H^s = \text{Sym}_2(E^*)
\end{align*}
\]
**Theorem (Weyman, Z. 2012)**

1. The bundle $p'^* \mathcal{T} \text{il}_K$ is a tilting bundle over $\mathcal{Z}^s$. In particular,

   $D^b(\text{Coh}(\mathcal{Z}^s)) \cong D^b(\text{End}_{\mathcal{Z}^s}(p'^* \mathcal{T} \text{il}_K)-\text{mod})$.

2. We have $\text{End}_{\mathcal{Z}^s}(p'^* \mathcal{T} \text{il}_K) \cong \text{End}_S(q'_* p'^* \mathcal{T} \text{il}_K)$, and it is a noncommutative desingularization of $\text{Spec } R^s$ (i.e., it has finite global dimension).

   Moreover, if $r = n - 1$, it is maximal Cohen-Macaulay over $R^s$. 

The quiver with relations

- The set of vertices is indexed by subpartitions of \( ((n - r)^r) \).
- The set of arrows from vertex labeled by partition \( \alpha \) to \( \beta \)

\[
\begin{cases}
(C^\alpha_{\beta,(1,0,\ldots,0)} E^*) \oplus (C^\beta_{\alpha,(1,0,\ldots,0)} E^*), & \text{if } n - r = 1 \\
(E^*) \oplus (C^\alpha_{\beta,(1,0,\ldots,0)} \mathbb{C}), & \text{if } n - r \geq 2
\end{cases}
\]

- The relations are generated by the subrepresentations in \( \text{Hom}(\beta, \alpha) \)

\[
\begin{align*}
(C^\alpha_{\beta,(1,1,0,\ldots,0)} \mathbb{S}_2 E^*) & \oplus (C^\beta_{\alpha,(1,1,0,\ldots,0)} \mathbb{S}_2 E^*) \oplus (\delta^\beta \wedge^2 E^*), & \text{if } n - r = 1 \\
(C^\beta_{\alpha',(1,-1)} \wedge^2 E^*) & \oplus (C^\beta_{\alpha',(-1,-2)} E^*) \oplus (C^\alpha_{\beta,(1,1,0,\ldots,0)} \mathbb{S}_2 E^*) \oplus (C^\beta_{\alpha,(2,0,\ldots,0)} \wedge^2 E^*), & \text{if } n - r = 2 \\
(C^\alpha_{\beta',(0,\ldots,0,-1,-1,-2)} \mathbb{C}) & \oplus (C^\beta_{\alpha',(1,0,\ldots,0,-1,-1)} E^*) \oplus (C^\beta_{\alpha',(2,0,\ldots,0)} \mathbb{S}_2 E^*) \oplus (C^\beta_{\alpha',(1,1,0,\ldots,0)} \wedge^2 E^*), & \text{if } n - 2 \geq 2
\end{align*}
\]
An example

We assume $r = n - 1$, $S^s = \mathbb{C}[x_{i,j}]_{1 \leq i \leq j \leq n}$ and $R^s = S^s/(\det(x_{ij}))$. Then $R^s$ is Gorenstein (since it’s a hypersurface).

The quiver with relations:

\[
\begin{array}{cccc}
\alpha_0(E) & \alpha_1(E) & \alpha_{n-1}(E) \\
\bullet_0 & \bullet_1 & \cdots & \bullet_{n-1} \\
\beta_0(E) & \beta_1(E) & \beta_{n-2}(E) \\
\end{array}
\]

relations:

\[\alpha_i \alpha_{i+1} (\wedge^2 E);\]

\[\beta_{i+1} \beta_i (\wedge^2 E);\]

\[(\beta_i \alpha_i + \alpha_{i+1} \beta_{i+1})(\mathcal{S}_2 E).\]
Another example: $n = \dim E = 4$ and $r = 2$

with relations:

$\text{Hom}(\begin{array}{c}
\end{array}, \emptyset) : \alpha_1 \alpha_3 (\wedge^2 E)$; $\text{Hom}(\begin{array}{c}
\end{array}, \emptyset) : \alpha_1 \alpha_2 (S_2 E)$;

$\text{Hom}(\begin{array}{c}
\end{array}, \emptyset) : \alpha_5 \alpha_6 (\wedge^2 E)$; $\text{Hom}(\begin{array}{c}
\end{array}, \emptyset) : \alpha_4 \alpha_6 (S_2 E)$;

$\text{Hom}(\begin{array}{c}
\end{array}, \emptyset) : \alpha_2 \alpha_4 - \alpha_3 \alpha_5 (S_2 E \oplus \wedge^2 E)$; $\text{Hom}(\begin{array}{c}
\end{array}, \emptyset) : \alpha_3 \beta_1 \alpha_1 - \alpha_3 \alpha_5 \beta_2 (\wedge^2 E)$;

$\text{Hom}(\begin{array}{c}
\end{array}, \emptyset) : \alpha_6 \beta_3 \alpha_5 - \beta_2 \alpha_3 \alpha_5 (\wedge^2 E)$; $\text{Hom}(\begin{array}{c}
\end{array}, \emptyset) : \beta_2 \alpha_3 \beta_1 - \alpha_6 \beta_3 \beta_1 (E)$;

$\text{Hom}(\begin{array}{c}
\end{array}, \emptyset) : \beta_3 \beta_1 \alpha_1 - \beta_3 \alpha_5 \beta_2 (E)$. 
Thank You!!!