Arc Diagram Varieties

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A report on a joint project with
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Contents

Short exact sequences of nilpotent linear operators
  The variety $V_{\alpha,\gamma}^\beta$
  Orbits correspond to arc diagrams
  Example

The stratification illustrated
  Properties
  Example, II: The arc diagrams
  Example, III: The embeddings

Two problems posed by Birge Huisgen-Zimmermann
  A negative answer
  A positive answer
  The extended bubble sort algorithm

The big picture
Definition: For $\alpha$ a partition and $k$ a field, we denote the nilpotent linear operator of type $\alpha$ by

$$N_\alpha = \bigoplus_i k[T]/(T^{\alpha_i}).$$
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For fixed partitions $\alpha$, $\beta$, $\gamma$, we are interested in short exact sequences of nilpotent linear operators

$$0 \longrightarrow N_\alpha \overset{f}{\longrightarrow} N_\beta \longrightarrow N_\gamma \longrightarrow 0$$
Short exact sequences of nilpotent linear operators

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\]

**Definition:** \( V^\beta_{\alpha, \gamma} = \{ f : N_\alpha \to N_\beta \mid \text{Cok}(f) \cong N_\gamma \} \)

The group \( G = \text{Aut}_{k[T]}(N_\alpha) \times \text{Aut}_{k[T]}(N_\beta) \) acts on \( V^\beta_{\alpha, \gamma} \), the orbits are in one-to-one correspondence with the equivalence classes of short exact sequences.
The stratification given by orbits

Proposition. Suppose all parts in the partition $\alpha$ are at most 2, i.e. $\alpha_1 \leq 2$. Then there is a one-to-one correspondence between the set of $G$-orbits in $\mathbb{V}_{\alpha,\gamma}^\beta$ and the set of arc diagrams of type $(\alpha, \beta, \gamma)$.

Definition: For an arc diagram $\Delta$ we denote the $G$-orbit in $\mathbb{V}_{\alpha,\gamma}^\beta$ corresponding to $\Delta$ by $\mathbb{V}_\Delta$. 
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**Definition:** For an arc diagram $\Delta$ we denote the $G$-orbit in $V_{\alpha,\gamma}^\beta$ corresponding to $\Delta$ by $V_\Delta$.

The subsets $V_\Delta$ define a stratification for $V_{\alpha,\gamma}^\beta$ in the following sense.

**Definition:** A family $(V_i)_{1 \leq i \leq n}$ of subsets of a variety $V$ is a stratification for $V$ if

1. Each $V_i$ is locally closed in $V$.
2. $V$ is the disjoint union $\bigcup_{1 \leq i \leq n} V_i$.
3. For each $i$ there is a subset $J_i \subset \{1, \ldots, n\}$ such that the closure $\overline{V_i}$ is just the union $\bigcup_{j \in J_i} V_j$. 

Arc diagrams

Assume that \( k \) is an algebraically closed field, and that \( \alpha, \beta, \gamma \) are partitions with \( \alpha_1 \leq 2 \). We consider the following two partial orderings on the set of arc diagrams of type \((\alpha, \beta, \gamma)\).
Arc diagrams

Assume that $k$ is an algebraically closed field, and that $\alpha$, $\beta$, $\gamma$ are partitions with $\alpha_1 \leq 2$. We consider the following two partial orderings on the set of arc diagrams of type $(\alpha, \beta, \gamma)$.

**Definition:** Denote by $\leq_{\text{arc}}$ the partial ordering on arc diagrams given by resolving intersections.
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**Definition:** Denote by $\leq_{\text{arc}}$ the partial ordering on arc diagrams given by resolving intersections.

**Definition:** Denote by $\leq_{\text{deg}}$ the partial ordering on arc diagrams given by $\Delta \leq_{\text{deg}} \Delta'$ if $\mathbb{V}_{\Delta'} \subset \overline{\mathbb{V}}_{\Delta}$. 
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**Definition:** Denote by $\leq_{\text{arc}}$ the partial ordering on arc diagrams given by resolving intersections.

![Diagram](image)

**Definition:** Denote by $\leq_{\text{deg}}$ the partial ordering on arc diagrams given by $\Delta \leq_{\text{deg}} \Delta'$ if $V_{\Delta'} \subset \overline{V}_{\Delta}$.

**Theorem (KS 2012):** $\Delta \leq_{\text{deg}} \Delta' \iff \Delta \leq_{\text{arc}} \Delta'$
Example: The variety $\mathcal{V}_{211,321}^{4321}$
Properties of the stratification

Theorem.

1. The strata $\mathbb{V}_\Delta$ are smooth locally closed subsets of dimension

$$\deg g_{\alpha, \gamma} + \deg a_{\alpha} - x(\Delta)$$

where

$$\begin{cases} 
\deg g_{\alpha, \gamma} = m(\beta) - m(\alpha) - m(\gamma) \\
\deg a_{\alpha} = |\alpha| + 2m(\alpha) \\
x(\Delta) = \#\{\text{intersections in } \Delta\} 
\end{cases}$$
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Theorem.

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2. *\( V_\Delta \subseteq V_{\Delta'} \) if and only if \( \Delta' \leq_{\text{arc}} \Delta \).*
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2. $\mathbb{V}_\Delta \subset \overline{\mathbb{V}}_{\Delta'}$ if and only if $\Delta' \leq_{\text{arc}} \Delta$.

3. There is a unique closed stratum. It has minimal dimension since it is given by the arc diagram with the largest number of intersections.
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4. There are exactly $c_{\alpha,\gamma}^\beta$ irreducible open strata, where $c_{\alpha,\gamma}^\beta$ is the leading coefficient of the Hall polynomial $g_{\alpha,\gamma}^\beta$. Those strata have maximal dimension since they correspond to the crossing-free arc diagrams.
The example revisited: The variety $\mathcal{V}_{211,321}^{4321}$
The example revisited: The variety \( V_{211,321}^{4321} \)

\[
\begin{align*}
\Delta_6 : & \quad \begin{array}{|c|c|}
\hline & 1 \\
\hline 1 & 1 \\
\hline 2 & 1 \\
\hline
\end{array} \\
\Delta_4 : & \quad \begin{array}{|c|c|}
\hline & 1 \\
\hline 1 & 1 \\
\hline 2 & 1 \\
\hline
\end{array} \\
\Delta_5 : & \quad \begin{array}{|c|c|}
\hline & 1 \\
\hline 2 & 1 \\
\hline 1 & 1 \\
\hline
\end{array} \\
\Delta_1 : & \quad \begin{array}{|c|c|}
\hline & 1 \\
\hline 1 & 1 \\
\hline 2 & 3 \\
\hline
\end{array} \\
\Delta_2 : & \quad \begin{array}{|c|c|}
\hline & 1 \\
\hline 2 & 2 \\
\hline 1 & 1 \\
\hline
\end{array} \\
\Delta_3 : & \quad \begin{array}{|c|c|}
\hline & 1 \\
\hline 1 & 2 \\
\hline 1 & 1 \\
\hline
\end{array}
\end{align*}
\]
The example revisited: The variety $\mathcal{V}_{211,321}^{4321}$
From short exact sequences to arc diagrams, I

Definition (Prüfer’s height sequence): Let \( v \in N_\beta \).

- \( \ell(v) = \min \{ n \in \mathbb{N}_0 : T^n v = 0 \} \)
- \( h(v) = \max \{ n \in \mathbb{N}_0 : v = T^n w \text{ for some } w \in N_\beta \}, \quad (v \neq 0) \)
- \( H(v) = (h(v), h(Tv), \ldots, h(T^{\ell(v)-1}v)) \).
From short exact sequences to arc diagrams, 1

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Example: Let $\beta = (4, 3, 2, 1)$, so $N_\beta = \{ f \in K[x, y] : \deg f \leq 4 \}$, $T = \frac{d}{dx}$. Assume $\text{char} K \neq 2, 3$. 

\[
\begin{array}{c}
\frac{1}{6}x^3 \\
\downarrow \\
\frac{1}{2}x^2 \\
\downarrow \\
x \\
\downarrow \\
1 \\
\end{array} 
\quad \begin{array}{c}
\frac{1}{2}x^2y \\
\downarrow \\
x y \\
\downarrow \\
y \\
\downarrow \\
y \\
\end{array} \quad \begin{array}{c}
xy^2 \\
\downarrow \\
y^2 \\
\downarrow \\
y^3 \\
\end{array}
\]
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$H(\frac{1}{6}x^3) = (1, 2, 3)$
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\[ H\left(\frac{1}{6}x^3\right) = (1, 2, 3) \]
\[ H\left(\frac{1}{2}x^2 y\right) = (1, 2) \]
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\begin{align*}
\frac{1}{6}x^3 &\quad H(\frac{1}{2}x^2) = (1, 2, 3) \\
1/2x^2 &\quad H(xy + y^2) = (1, 2) \\
x &\quad H(x + y^3) = (0, 3)
\end{align*}

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\begin{array}{c}
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1/2x^2 \\
x \\
y
\end{array}
\quad
\begin{array}{c}
\frac{1}{2}x^2y \\
xy \\
y \\
y^2 \\
y^3
\end{array}
\]
From short exact sequences to arc diagrams, II

Proposition: For partitions $\alpha, \beta, \gamma$ with $\alpha_1 \leq 2$, there is a one-to-one correspondence

$$\{\text{embeddings in } \mathbb{V}^\beta_{\alpha, \gamma}(K)\}/\sim \overset{1-1}{\longleftrightarrow} \{\text{arc diagrams of type } (\alpha, \beta, \gamma)\}.$$
Proposition: For partitions $\alpha, \beta, \gamma$ with $\alpha_1 \leq 2$, there is a one-to-one correspondence

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$$U_2 = \langle 1, y, xy^2, y^2 \rangle$$
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Two problems posed by Birge Huisgen-Zimmermann

Consider a stratification \((V_i)\) of a variety \(V\).

**Definition:** A chain \(V' = V_{i_0} <_{\text{deg}} V_{i_1} <_{\text{deg}} \cdots <_{\text{deg}} V_{i_s} = V''\) of strata is **saturated** if it has no refinement.
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**Question 1:** Suppose \(V' \preceq_{\text{deg}} V''\). Do all saturated chains have the same length?
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**Question 1:** Suppose \(V' \leq_{\text{deg}} V''\). Do all saturated chains have the same length?

**Question 2:** Suppose \(V' \leq_{\text{deg}} V''\). Is there a chain such that subsequent strata have dimension difference equal to one?
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**Question 2:** Suppose \(V' \leq_{\deg} V''\). Is there a chain such that subsequent strata have dimension difference equal to one?

In general, the answer to Question 2 is NO. Consider the example:
A negative answer

Recall Birge’s first question:

**Question 1:** Suppose $V' \leq_{\text{deg}} V''$. Do all saturated chains have the same length?

In general, this is not the case.
A negative answer

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A positive answer

Recall Birge’s questions:

**Question 1:** Suppose $V' \leq_{\text{deg}} V''$. Do all saturated chains have the same length?

**Question 2:** Suppose $V' \leq_{\text{deg}} V''$. Is there a chain such that subsequent strata have dimension difference equal to one?

**Proposition.** Suppose $\beta \setminus \gamma$ is a vertical strip. For any stratum $V''$ there is a stratum of maximal dimension $V'$ and a saturated chain which is such that subsequent strata have dimension difference one.
The extended bubble sort algorithm

(extended bubble sort)
Repeat (1) - (3):

(1) Starting from the right, select a maximal set of arcs and poles such that every source is on the left of every target.

(2) (standard bubble sort)
Repeat (a) - (c):
    (a) Start at the left.
    (b) For each two neighboring targets of selected arcs and poles, if they intersect, exchange the sources.
    (c) If possible, move to the right. Go to (b).

Until there are no crossings between the selected arcs and poles.

(3) Remove arcs between neighboring points, until none are left.

Until there are no crossings left.
The extended bubble sort algorithm by example

...or how to resolve crossings one-by-one:
The extended bubble sort algorithm by example

...or how to resolve crossings one-by-one: 8-
The extended bubble sort algorithm by example

...or how to resolve crossings one-by-one: 8- 7-
The extended bubble sort algorithm by example

...or how to resolve crossings one-by-one: 8- 7- 6-
The extended bubble sort algorithm by example

...or how to resolve crossings one-by-one: 8- 7- 6- 5-
The extended bubble sort algorithm by example

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The extended bubble sort algorithm by example

...or how to resolve crossings one-by-one: 8- 7- 6- 5- 4-
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The big picture

Consider the embedding problems:

- Projective Varieties $\mathbb{P}(V)$: Embeddings $K \to V$
  $\text{Gl}(V)$ acts transitively
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- Subgroups of $p^n$-bounded abelian groups:
  Embeddings $H \to G$
  Birkhoff’s Problem: Classify $\text{Aut}H \times \text{Aut}G$-orbits
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- Subgroups of $p^n$-bounded abelian groups:
  Embeddings $H \to G$
  Birkhoff’s Problem: Classify $\text{Aut}H \times \text{Aut}G$-orbits
- Invariant subspaces of nilpotent linear operators:
  Embeddings $N_\alpha \to N_\beta$
  $G$-orbits define a stratification for $\nabla^\beta_{\alpha, \gamma}$ in which
    - orbit dimensions,
    - degeneration relation,
    - structure of the poset given by $\leq_{\text{deg}}$
    - saturation properties of chains in this poset
  are controlled by arc diagrams.