Universal deformation rings and tame blocks with two simple modules.

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In the 1980’s, Mazur, using work of Schlessinger, introduced deformations of Galois representations to study lifts of Galois representations over finite fields to p-adic representations.
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The main motivation for determining universal deformation rings for finite groups is to test or verify conjectures about the ring structure of universal deformation rings for arbitrary Galois groups.
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Determine all finitely generated $kG$-modules $V$ which belong to $B$ and whose endomorphism ring is isomorphic to $k$. 
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Determine all finitely generated $kG$-modules $V$ which belong to $B$ and whose endomorphism ring is isomorphic to $k$.

First calculate the universal deformation ring modulo 2 and then calculate the universal deformation ring $R(G, V)$ for each of these modules.
Definitions and Background

Let $k$ be an algebraically closed field of positive characteristic $p$.

Let $G$ be a finite group.

Let $kG$ be the group algebra of $G$ with coefficients in $k$.

All modules will be finitely generated left modules.
Universal Deformation Rings

Let $V$ be a finitely generated $kG$-module.

Let $\hat{\mathcal{C}}$ be the category of all complete local commutative Noetherian rings with residue field $k$ where the morphisms are the homomorphisms of complete local rings inducing the identity on $k$.

Let $R \in Ob(\hat{\mathcal{C}})$. 

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Definition:

(i) A lift of $V$ over $R$ is a finitely generated $RG$-module $M$ which is free over $R$ together with a $kG$-module isomorphism $\phi : k \otimes_R M \rightarrow V$. Notation: $(M, \phi)$.
Definition:

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(ii) Two lifts $(M, \phi)$ and $(M', \phi')$ are isomorphic if there exists an $RG$-module isomorphism $f : M \rightarrow M'$ with $\phi' \circ (id \otimes f) = \phi$.
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(ii) Two lifts $(M, \phi)$ and $(M', \phi')$ are isomorphic if there exists an $RG$-module isomorphism $f : M \longrightarrow M'$ with $\phi' \circ (id \otimes f) = \phi$.

(iii) A deformation of $V$ over $R$ is an isomorphism class of a lift $(M, \phi)$ of $V$ over $R$. Notation: $[(M, \phi)]$. 

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UDRs and tame blocks with two simple modules
Theorem: (Mazur; Bleher-Chinburg) Suppose $\text{End}_{kG}(V) \cong k$. Then there exists an $R(G, V)$ in $\hat{C}$ and a lift $(U(G, V), \phi_U)$ of $V$ over $R(G, V)$ such that for all $A \in \hat{C}$ and for all lifts $(M, \phi)$ of $V$ over $A$, there exists a unique $\alpha : R(G, V) \longrightarrow A$ in $\hat{C}$ such that the lift $(M, \phi)$ is isomorphic to the lift $(A \otimes_{R(G, V)} \alpha U(G, V), \phi')$ where $\phi'$ is the composition $k \otimes_A (A \otimes_{R(G, V)} \alpha U(G, V)) \cong k \otimes_{R(G, V)} U(G, V) \xrightarrow{\phi_U} V$. The pair $(R(G, V), [(U(G, V), \phi_U)])$ is unique up to isomorphism.
Definition:

The ring $R(G, V)$ is called the universal deformation ring of $V$.

$[(U(G, V), \phi_U)]$ is called the universal deformation of $V$ over $R(G, V)$. 
Let \( W = W(k) \) be the ring of infinite Witt vectors over \( k \).

Since \( k \) is assumed to be algebraically closed of characteristic \( p \), \( W \) is the unique (up to isomorphism) complete discrete valuation ring with residue field \( k \) such that \( p \) generates the maximal ideal.

Ex. \( k = \mathbb{F}_p \) implies \( W = \text{completed } p\text{-adic integers} \)
Theorem: (Mazur)

(i) If $\dim_k(\text{Ext}_{kG}^1(V, V)) = m$, then there exists a surjective homomorphism $\Phi : W[[t_1, t_2, \ldots, t_m]] \twoheadrightarrow R(G, V)$ in $\hat{C}$, and $m$ is minimal with this property.

(ii) If $\dim_k(\text{Ext}_{kG}^2(V, V)) = s$, then $s$ is an upper bound for the minimal number of generators for $\text{Ker}(\Phi)$.
Theorem: (Mazur)

(i) If $\dim_k(\text{Ext}^1_{kG}(V, V)) = m$, then there exists a surjective homomorphism $\Phi : W[[t_1, t_2, ..., t_m]] \to R(G, V)$ in $\hat{C}$, and $m$ is minimal with this property.

(ii) If $\dim_k(\text{Ext}^2_{kG}(V, V)) = s$, then $s$ is an upper bound for the minimal number of generators for $\text{Ker}(\Phi)$. 

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UDRs and tame blocks with two simple modules
For the remainder of this talk, we assume \( \text{char}(k) = 2 \).

Let \( G \) be a finite group and let \( B \) be a block of the group algebra \( kG \) such that \( B \) has semi-dihedral or generalized quaternion defect groups and precisely two isomorphism classes of simple modules.

Then \( B \) is Morita equivalent to the algebra \( SD(2A)_1, SD(2A)_2, SD(2B)_1, SD(2B)_2 \), or \( SD(2B)_3 \) if \( B \) has semi-dihedral defect groups or to the algebra \( Q(2A), Q(2B)_1, \) or \( Q(2B)_2 \) if \( B \) has generalized quaternion defect groups. [Erdmann]
**SEMI-DIHEDRAL DEFECT GROUPS**

Let $SD(2A)_1$ be the finite dimensional $k$-algebra with quiver

\[
Q = \begin{array}{cc}
\alpha & \beta \\
0 & \gamma & 1
\end{array}
\]

and relations

\[
\alpha^2 = c(\gamma \beta \alpha)^t, \quad \beta \gamma \beta = \beta \alpha (\gamma \beta \alpha)^{t-1}, \\
\gamma \beta \gamma = \alpha \gamma (\beta \alpha \gamma)^{t-1}, \text{ and } \alpha (\gamma \beta \alpha)^t = 0
\]

where $t \geq 2$, $t = 2^{n-1}$ and $c \in k$. 

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UDRs and tame blocks with two simple modules
There are two simple $SD(2A)_1$-modules corresponding to the vertices 0 and 1 which we denote by $S_0$ and $S_1$, and there are two indecomposable projective $SD(2A)_1$-modules up to isomorphism, which can be described using the following diagrams:

$$P_0 = \begin{array}{c}
0 \\
0 \\
1 \\
0 \\
1 \\
0 \\
0 \\
1 \\
0 \\
0
\end{array} \quad \text{and} \quad P_1 = \begin{array}{c}
1 \\
0 \\
1 \\
0 \\
1 \\
0 \\
0 \\
1 \\
0 \\
0
\end{array}$$

where the line $*$ and $**$ in $P_0$ corresponds to the relation $\alpha^2 = c(\gamma\beta\alpha)^t$ and $\beta\gamma\beta = \beta\alpha(\gamma\beta\alpha)^{t-1}$, respectively.
Let $SD(2A)_2$ be the finite dimensional $k$-algebra with quiver

$$Q = \begin{array}{c}
\bullet \\
\alpha \leftarrow \beta \\
\gamma \\
0 \quad 1
\end{array}$$

and relations

$$\alpha^2 = \gamma \beta (\alpha \gamma \beta)^{t-1} + c(\gamma \beta \alpha)^t, \ (\alpha \gamma \beta)^t = (\gamma \beta \alpha)^t, \text{ and } \beta \gamma = 0$$

where $t \geq 2$, $t = 2^{n-2}$ and $c \in k$. 
There are two simple $SD(2A)_2$-modules corresponding to the vertices 0 and 1 which we denote by $S_0$ and $S_1$, and there are two indecomposable projective $SD(2A)_2$-modules up to isomorphism, which can be described using the following diagrams:

\[
\begin{array}{c}
\begin{array}{c}
0 \\
0 \\
1 \\
* \\
0 \\
0 \\
0 \\
1 \\
0 \\
1 \\
0
\end{array}
\begin{array}{c}
0 \\
1 \\
0 \\
1 \\
0 \\
0 \\
1 \\
0 \\
1 \\
0
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{c}
0 \\
0 \\
1 \\
0 \\
0 \\
0 \\
1 \\
0 \\
1 \\
0
\end{array}
\begin{array}{c}
1 \\
0 \\
0 \\
1 \\
0 \\
0 \\
1 \\
0 \\
0 \\
0
\end{array}
\end{array}
\]

where the line $*$ in $P_0$ corresponds to the relation
\[\alpha^2 = \gamma\beta(\alpha\gamma\beta)^{t-1} + c(\gamma\beta\alpha)^t.\]
Let $SD(2B)_1$ be the finite dimensional $k$-algebra with quiver

$$Q = \begin{array}{c}
\circlearrowleft \alpha \rightarrow \bullet \rightarrow \beta \\
\circlearrowleft \circlearrowleft \gamma \leftarrow \eta \\
\end{array}$$

and relations

$$\eta \beta = 0 = \gamma \eta = \beta \gamma, \quad \alpha^2 = \gamma \beta + c(\gamma \beta \alpha), \quad \gamma \beta \alpha = \alpha \gamma \beta, \quad \beta \alpha \gamma = \eta^t$$

where $t \geq 2$, $t = 2^{n-2}$, and $c \in k$. 
There are two simple $SD(2B)_1$-modules corresponding to the vertices 0 and 1 which we also denote by $S_0$ and $S_1$, and there are two indecomposable projective $SD(2B)_1$-modules up to isomorphism, which can be described using the following diagrams:

$$P_0 = \begin{array}{ccc}
0 & \ast & 1 \\
1 & 0 & \multicolumn{1}{c}{} \\
0 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{}
\end{array} \quad \text{and} \quad P_1 = \begin{array}{ccc}
0 & 1 & \\
0 & 1 & \multicolumn{1}{c}{} \\
0 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{}
\end{array}$$

where the line $\ast$ in $P_0$ corresponds to the relation $\alpha^2 = \gamma\beta + c(\gamma\beta\alpha)$. 

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Let $SD(2B)_2$ be the finite dimensional $k$-algebra with quiver

$$Q = \begin{array}{c}
\circlearrowleft \\
0 \leftrightarrow 1
\end{array} \quad \begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\eta
\end{array}$$

and relations

$$
\eta^2 \beta = 0 = \gamma \eta^2, \quad \alpha^2 = c(\gamma \beta \alpha)^2, \quad \beta \gamma = \eta^{t-1},
\gamma \eta = \alpha \gamma (\beta \alpha \gamma), \quad \eta \beta = \beta \alpha (\gamma \beta \alpha)
$$

where $t \geq 4$, $t = 2^{n-2}$, and $c \in k$. 

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UDRs and tame blocks with two simple modules
There are two simple $SD(2B)_2$-modules corresponding to the vertices 0 and 1 which we also denote by $S_0$ and $S_1$, and there are two indecomposable projective $SD(2B)_2$-modules up to isomorphism, which can be described using the following diagrams:

\[ P_0 = \begin{array}{c}
0 \\
0 \\
1 \\
0 \\
0 \\
0 \\
1 \\
0 \\
1
\end{array} \quad \text{and} \quad P_1 = \begin{array}{c}
0 \\
0 \\
1 \\
0 \\
0 \\
0 \\
1 \\
0 \\
1
\end{array} \]

where the lines $\ast$ and $\ast\ast$ in $P_0$ corresponds to the relations $\alpha^2 = c(\gamma\beta\alpha)^2$ and $\eta\beta = \beta\alpha(\gamma\beta\alpha)$ and the lines $\bullet$ and $\bullet\bullet$ in $P_1$ corresponds to the relations $\beta\gamma = \eta^{t-1}$ and $\gamma\eta = \alpha\gamma(\beta\alpha\gamma)$, respectively.
Let $SD(2B)_3$ be the finite dimensional $k$-algebra with quiver

$$Q = \begin{array}{c}
\alpha \circlearrowleft \\
\downarrow \beta \\
\circlearrowright \eta \\
\downarrow \gamma \\
0 \\
\end{array}$$

and relations

$$
\begin{align*}
\alpha^2 &= \gamma \beta, \\
\beta \alpha &= \eta \beta, \\
\gamma \eta &= \alpha \gamma, \\
\beta \gamma &= \eta^2 + c \eta^{t+1}, \\
\beta \alpha^t &= \alpha^t \gamma = \alpha^{t+2} = \eta^{t+2} = 0, \\
\gamma \eta^t &= \eta^t \beta = 0
\end{align*}
$$

where $t \geq 2$, $t = 2^{n-2} - 1$, and $c \in k$ [Holm].
There are two simple $SD(2B)_3$-modules corresponding to the vertices 0 and 1 which we also denote by $S_0$ and $S_1$, and there are two indecomposable projective $SD(2B)_3$-modules up to isomorphism, which can be described using the following diagrams:

$$
P_0 = \begin{array}{c}
0 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
\cdots & \cdots & \cdots \\
0 & 0 & 1 \\
0 & 1
\end{array}$$

and

$$
P_1 = \begin{array}{c}
1 \\
0 & 0 & * & 1 \\
0 & 0 & 1 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & 1 \\
0 & 1
\end{array}$$

where the line $*$ in $P_1$ corresponds to the relation $\beta \gamma = \eta^2 + c\eta^{t+1}$. 

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GENERALIZED QUATERNION DEFECT GROUPS

Let $Q(2A)$ be the finite dimensional $k$-algebra with quiver

$$Q = \begin{array}{c}
\circ \xrightarrow{\alpha} \circ \xleftarrow{\beta} \circ \\
0 \xleftarrow{\gamma} 1
\end{array}$$

and relations

$$\alpha^2 = \gamma \beta (\alpha \gamma \beta)^{s-1} + c(\alpha \gamma \beta)^s, \quad \beta \gamma \beta = \beta \alpha (\gamma \beta \alpha)^{s-1},$$

$$\gamma \beta \gamma = \alpha \gamma (\beta \alpha \gamma)^{s-1}, \quad \beta \alpha^2 = 0$$

where $s \geq 2$, $s = 2^{n-1}$ and $c \in k$. 

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UDRs and tame blocks with two simple modules
There are two simple $Q(2A)$-modules corresponding to the vertices 0 and 1 which we denote by $S_0$ and $S_1$, and there are two indecomposable projective $Q(2A)$-modules up to isomorphism, which can be described using the following diagrams:

$$P_0 = \begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{array}$$

and

$$P_1 = \begin{array}{cccc}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{array}$$

where the line * and ** in $P_0$ corresponds to the relations

$$\alpha^2 = \gamma \beta (\alpha \gamma \beta)^{s-1} + c (\alpha \gamma \beta)^s \quad \text{and} \quad \beta \gamma \beta = \beta \alpha (\gamma \beta \alpha)^{s-1},$$

respectively.
Let $Q(2B)_1$ be the finite dimensional $k$-algebra with quiver

$$Q = \begin{array}{c}
\bullet \\
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\eta
\end{array}
\end{array}
$$

and relations

$$\alpha^2 = \gamma \beta (\alpha \gamma \beta) + c (\alpha \gamma \beta)^2, \ \eta \beta = \beta \alpha (\gamma \beta \alpha),$$

$$\beta \gamma = \eta^{s-1}, \ \gamma \eta = \alpha \gamma (\beta \alpha \gamma), \ \beta \alpha^2 = 0$$

where $s \geq 4$, $s = 2^{n-2}$, and $c \in k$. 
There are two simple $Q(2B)_1$-modules corresponding to the vertices 0 and 1 which we also denote by $S_0$ and $S_1$, and there are two indecomposable projective $Q(2B)_1$-modules up to isomorphism, which can be described using the following diagrams:

\[
P_0 = \begin{array}{cccc}
0 & \ast & \ast & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
\end{array}
\quad \text{and} \quad
P_1 = \begin{array}{cccc}
0 & \bullet & \bullet & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
\end{array}
\]

where the lines $\ast$ and $\ast\ast$ in $P_0$ corresponds to the relations $\alpha^2 = \gamma\beta(\alpha\gamma\beta) + c(\alpha\gamma\beta)^2$ and $\eta\beta = \beta\alpha(\gamma\beta\alpha)$ and the lines $\bullet$ and $\bullet\bullet$ in $P_1$ corresponds to the relations $\beta\gamma = \eta^{s-1}$ and $\gamma\eta = \alpha\gamma(\beta\alpha\gamma)$, respectively.
Let $Q(2B)_2$ be the finite dimensional $k$-algebra with quiver

$$Q = \begin{array}{c}
\bullet \\
\alpha
\end{array} \xrightarrow{\beta} \begin{array}{c}
\bullet \\
\eta
\end{array} \xleftarrow{\gamma} \begin{array}{c}
\bullet \\
\gamma
\end{array}$$

and relations

$$\beta\alpha = \eta\beta, \quad \gamma\eta = \alpha\gamma, \quad \gamma\beta = \alpha^2 + \alpha^3 q(\alpha),$$
$$\beta\gamma = \eta^2 + \eta^3 q(\eta) + a\eta^{s-1} + c\eta^s, \quad \alpha^{s+1} = 0 = \eta^{s+1},$$
$$\alpha^{s-1}\gamma = 0 = \beta\alpha^{s-1}$$

where $q(t) \in k[t], \ s \geq 4, \ s = 2^{n-2}, \ a, c \in k$ and $a \neq 0$. 
There are two simple $Q(2B)_2$-modules corresponding to the vertices 0 and 1 which we also denote by $S_0$ and $S_1$, and there are two indecomposable projective $Q(2B)_2$-modules up to isomorphism, which can be described using the following diagrams:

$P_0 = \begin{array}{cccc}
0 & \star & 1 \\
0 & 0 & 1 \\
\vdots & \vdots & \vdots \\
0 & 0 & 1 \\
0 & 0 & \end{array}$ \quad \text{and} \quad
P_1 = \begin{array}{cccc}
0 & \bullet & 1 \\
0 & 0 & 1 \\
\vdots & \vdots & \vdots \\
0 & 0 & 1 \\
0 & 0 & \end{array}$

where the line $\star$ in $P_0$ and $\bullet$ in $P_1$ corresponds to the relations

$\gamma \beta = \alpha^2 + \alpha^3 q(\alpha)$ and $\beta \gamma = \eta^2 + \eta^3 q(\eta) + a \eta^{s-1} + c \eta^s$,

respectively.
Modules with Endomorphism Rings Isomorphic to $k$

**Theorem (BLS):** Let $\Lambda$ be $Q(2A)$. Let $M$ be a $\Lambda$-module. Then $\text{End}_\Lambda(M) \cong k$ if and only if

$$M \in \{ S_0, S_1, \begin{array}{cc} S_0 & S_1 \\ S_1 & S_0 \end{array}, \begin{array}{cc} S_0 & S_1 \\ S_0 & S_0 \end{array}, \begin{array}{cc} S_0 & S_1 \\ S_1 & S_0 \end{array} \}.$$
Sketch of proof:

Case (i) Radical length is 1: $S_0, S_1$
Sketch of proof:

Case (i) Radical length is 1: \( S_0, S_1 \)

Case (ii) Radical length is 2: 

\[
\begin{array}{ccccccccc}
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0' & 1' & 0' & 0 & 1' & 0' & 0 & 1 \\
\end{array}
\]
Case(iii) Radical length is 3:
Case(iii) Radical length is 3:

a) Suppose $\text{top}(M)$ is a direct sum of copies of $S_1$:

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
0, 0, 0, 1, 0, 1, 0 & 0, 1, 0, 0, 1, 0, 1 \\
0, 1, 0, 0, 1, 0, 1, 0, 1 & 0, 1, 0, 0, 1, 0, 1, 0, 1
\end{array}
\]
b) Suppose $\text{top}(M)$ is a direct sum of copies of $S_0$: $0, 0, 1, 1$.
Case(iv) Radical length is greater than 3: Consider $M / \text{rad}^3(M)$ and $\text{soc}_3(M)$.

a) Suppose $\text{top}(M)$ is a direct sum of copies of $S_1$ and $\text{soc}(M)$ is a direct sum of copies of $S_0$.

b) Suppose $\text{top}(M)$ is a direct sum of copies of $S_0$ and $\text{soc}(M)$ is a direct sum of copies of $S_1$. 
Case (iv) Radical length is greater than 3: Consider $M/\text{rad}^3(M)$ and $\text{soc}_3(M)$.

a) Suppose $\text{top}(M)$ is a direct sum of copies of $S_1$ and $\text{soc}(M)$ is a direct sum of copies of $S_0$. 
Case (iv) Radical length is greater than 3: Consider $M/\text{rad}^3(M)$ and $\text{soc}_3(M)$.

a) Suppose $\text{top}(M)$ is a direct sum of copies of $S_1$ and $\text{soc}(M)$ is a direct sum of copies of $S_0$.

b) Suppose $\text{top}(M)$ is a direct sum of copies of $S_0$ and $\text{soc}(M)$ is a direct sum of copies of $S_1$. 