FINE/COARSE MODULI SPACES

IN THE REPRESENTATION THEORY

OF FINITE DIMENSIONAL ALGEBRAS

A basic finite dim'l algebra \( k = \overline{k} \)

w.l.o.g., \( \Lambda = \overline{k} Q / I \) for a quiver \( Q \).

Vertices: \( Q_0 = \{ e_1, \ldots, e_n \} \),
identified w. primitive idempotents of \( \Lambda \).

\( J = \) Jacobson radical of \( \Lambda \),
\( S_i = \Lambda e_i \overline{J} e_i \) the simples in \( \Lambda\text{-mod} \)
WERE RIEMANN AROUND TO CONSULT US, THIS IS WHAT HE'D SAY (I THINK)

**Starting point:** \( \mathcal{C} \subseteq \Lambda\)-mod
class of objects

(1) Identify "suitable" discrete invariants of the modules in \( \mathcal{C} \) to subdivide \( \mathcal{C} \) into subclasses \( \mathcal{C}_i \).

(2) For each \( i \), find a variety \( V_i \) and a bijection

\[ V_i \leftrightarrow \{ \text{iso classes in } \mathcal{C}_i \} \]

which "continuously" parametrizes the iso classes in \( \mathcal{C}_i \).

Hope:
- Geometry of \( V_i \) should reflect structural changes of modules in \( \mathcal{C}_i \).
- Parametrization should be "universal" in a suitable sense.
Tentative parametrizations of the \( \Lambda \)-modules w. dim vector \( d = (d_1, \ldots, d_n) \)

(I) classical affine: \( Q_1 = \) set of arrows.

\[
\text{Rep}(\Lambda, d) = \left\{ x = (x_{ij})_{i,j \in Q} \mid x_{ij} \in \text{Hom}_K(K^{d_i}, K^{d_j}) \text{ for } \alpha : i \to j, \text{ the } x_{ij} \text{ satisfy the relations in } I \right\}
\]

Have morphic action of the reductive algebraic group \( GL(d) = \prod_{i \in \mathbb{Z}} GL(d_i) \).

\[\text{1-1 correspondence} \]

\[
\{ \text{GL}(d) \text{-orbits of } \text{Rep}(\Lambda, d) \} \leftrightarrow \left\{ \text{iso classes of } M \in \Lambda \text{-mod w. dim } M = d \right\}
\]
Projective parametrization of the same objects

\( P = \text{projective cover of } \bigoplus_{i \in \mathbb{N}} S_i^{d_i} \),

\( d' = \dim P - 1 \cdot d \)

\[ \text{GRASS}(\Lambda, d) = \left\{ C \in \text{Gr}(d', P) \mid \begin{array}{c}
\forall C \in \bigwedge \text{submodule} \\
\dim P/C = d
\end{array} \right\} \]

a projective variety.

Obvious action of the algebraic group \( \text{Aut}_\Lambda(P) \) : \( g \cdot C = g(C) \)

\( \rightarrow 1-1 \) correspondence

\[ \left\{ \text{Aut}_\Lambda(P)\text{-orbits of GRASS}(\Lambda, d) \right\} \leftrightarrow \left\{ \text{iso classes of } M \in \Lambda\text{-mod } \dim M = d \right\} \]

\( \text{Aut}_\Lambda(P). C \mapsto P/C \)
Situation: Linear algebraic group \( G \) acts morphically on \( V \) (variety).

Categorical quotient of \( V \) by \( G \):
A morphism \( \phi : V \to V//G \)
which is constant on \( G \)-orbits, such that any morphism \( V \to \text{variety} \)
with this property factors uniquely through \( \phi \).

Orbit space: \( \phi : V \to V//G \)
as before such that
fibers of \( \phi = G \)-orbits of \( V \).

Geometric quotient: orbit space \( \phi \), open morphism such that, for
open \( U \subseteq V//G \), \( \mathcal{O}(\phi^{-1}(U))^G \cong \mathcal{O}(U) \).

naturally
AN IDEA: It looks hopeless, but has potential at 2nd (or 3rd) glance.

Factor the group action out of the parametrizing variety to obtain a variety whose points are in bijection with the iso classes of modules we want to classify.

**Major cliff:** This results in a (Zariski) topological quotient but not in a variety, in general.
COMPARING OUR PARAMETRIZATIONS

Good transfer of geometric info
[Bongartz - H Z]:

\[
\begin{align*}
&\{\text{GL}(d)\text{-stable subsets of} \ \text{Rep}(\Lambda, d)\} \\
&\leftrightarrow \{\text{Aut}_\Lambda(P)\text{-stable subsets of} \ \text{GRASS}(\Lambda, d)\} \\
&\text{(GL}(d)\text{-orbit of) } \\
&\text{(Aut}_\Lambda(P)\text{-orbit of) } \\
&\text{(module M) } \\
&\text{(the same M) }
\end{align*}
\]

Suppose \( V \subseteq \text{Rep}(\Lambda, d) \) is a \( \text{GL}(d) \)-stable subvariety, and \( W = \psi(V) \) the corresponding subvariety of \( \text{GRASS} \).

\[ \exists \text{Rep}(\Lambda, d)/\text{GL} \leftrightarrow \exists \text{GRASS}(\Lambda, d)/\text{Aut} \]

orbit space \leftrightarrow \text{orbit space}

geometric quotient \leftrightarrow \text{geometric quotient}
WE KNOW WHAT WE WANT:

$GL(d)$-stable subvariety $V$ of $Rep(\Lambda, d)$ (resp. $Aut(x)$-stable subvariety $V$ of $\text{GRASS}(\Lambda, d)$) which admits (at least) an orbit space modulo the pertinent action.

**CRUX:** (Relatively) closed orbits are a necessary condition for this wish to come true.
(I) **Affine parametrization**

$GL(d)$ is a reductive group acting on the affine variety $Rep(\Lambda, d)$. Hence: If $R = \text{coordinate ring of } Rep(\Lambda, d)$, then the invariant ring $R^{GL}$ is a fin. gen. $K$-algebra & 

$\text{Spec}(R^{GL}) = \text{Rep}(\Lambda, d) // GL(d)$.

Looks better than it is: $R^{GL} \cong K$, so $\text{Spec}(R^{GL})$ is a singleton.

Trouble: $Rep(\Lambda, d)$ contains only a single closed orbit.
(II) Projective parametrization

Acting group:
\[ \text{Aut}_\Lambda(P) \cong \text{Aut}_\Lambda(P/JP) \times U, \]
where \( U = \{ \text{id} + f \mid f \in \text{Hom}_\Lambda(P, JP) \} \) is a unipotent subgroup.

Rosenlicht: Action of a unipotent group on an affine variety has closed orbits.

How to benefit? \exists U-stable affine cover of GRASS(\( \Lambda, d \)), which is representation-theoretically defined.

This fact will guide our hand in selecting amenable Aut_\Lambda(P)-stable subvarieties of GRASS(\( \Lambda, d \)).
"Continuity" of a parametrization of modules by a variety is made precise by the concept of a "family".

**DEF.** [King] A family of \( L \)-modules with dim-vector \( d \), parametrized by a variety \( X \), is a rank-1 \( d \)\( \times 1 \) vector bundle \( \Delta \) over \( X \) together with a \( k \)-algebra homomorphism \( s: \Lambda \rightarrow \text{End}(\Delta) \).

**Family induced from** \((\Delta, s)\): Let \( \tau: Y \rightarrow X \) be a morphism, \( \tau^*(\Delta) \) the pullback bundle. The bundle \( \tau^*(\Delta) \) over \( Y \) comes with induced alg. homomorphism \( \Lambda \rightarrow \text{End}(\tau^*(\Delta)) \).
FINN: A fine moduli space for a class $C$ of $d$-dim'l $\Lambda$-modules is a variety $X$ that parametrizes a family $\Delta$ of modules in $C$ endowed with the following universal property:

Whenever $\Delta'$ is a family of modules in $C$, parametrized by $X'$ say, \( \exists ! \) morphism $\tau: X' \to X$ with $\Delta' \cong \tau^*(\Delta)$

means: $\forall x' \in X'$, the fiber of $\Delta'$ above $x'$ is isomorphic, as a $\Lambda$-module, to the fiber of $\tau^*(\Delta)$ above $x'$. 
COARSE: Still $\mathcal{C} \subseteq \Lambda$-mod.

Let $\text{Rep}(\mathcal{C})$ be the union of the $\text{GL}(d)$-orbits in $\text{Rep}(\Lambda, d)$ of the modules in $\mathcal{C}$.

Let $\text{GRASS}(\mathcal{C})$ be the union of the $\text{Aut}_\Lambda(P)$-orbits of these modules in $\text{GRASS}(\Lambda, d)$.

A variety $X$ is a coarse moduli space for $\mathcal{C}$ if

$$X \cong \text{Rep}(\mathcal{C})//\text{GL}(d)$$

is an orbit space;

or equivalently, if

$$X \cong \text{GRASS}(\mathcal{C})//\text{Aut}_\Lambda(P)$$

is an orbit space.
DICHOTOMY OF METHODS IN
RESTRICTING TO CLASSES $C \subseteq \text{mod}$
WHICH ADMIT MODULI SPACES

(I) Working with $\text{Rep}(\Lambda, \delta)$

King's adaptation of GIT

Idea: Deal with lack of closed orbits by passing to subvarieties of $\text{Rep}(\Lambda, \delta)$ in which more orbits become closed.

This works best for path algebras without relations, so let us assume $\Lambda = \kappa \mathbb{Q}$. 
Some Detail on King's Approach

If $\Lambda = K Q$, then $\text{Rep}(\Lambda, \Omega)$ is a finite dimensional vector space, and the $\text{GL}(\Omega)$-action is linear. Let $R$ be the coordinate ring of $\text{Rep}(\Lambda, \Omega)$ and $\chi: \text{GL}(\Omega) \to K$ a character (identifiable with a map $Q_0 \to \mathbb{Z}$).

The $\chi$-semi-invariants:

$R_\chi^\infty := \{ f \in R \mid f(g, x) = \chi(g) f(x) \forall x, g \}$

Replace $\text{Spec}(R G)$ by $\text{Proj}$ of the graded ring $\bigoplus_{n \geq 0} R_\chi^n$, and cut down $\text{Rep}(\Lambda, \Omega)$ to

$\text{Rep}^\text{ss} = \{ x \in \text{Rep}(\Lambda, \Omega) \mid \exists n \geq 1, f \in R_\chi^n \text{ with } f(x) \neq 0 \}$

the $\Theta$-semistable points
King's GIT-Based Theorem

Given \( \Theta : \mathbb{Q}_0 \to \mathbb{Z} \) (extend to \( \mathbb{Z} \mathbb{Q}_0 \)). This yields a character \( \chi \) of \( \text{GL}(d) \).

**Theorem.** Part 1. Suppose \( \Theta(d) = 0 \).

- The \( \chi \)-semistable points in \( \text{Rep}(\Lambda, d) \) correspond to the modules \( M \) (\( \Theta \)-semistable) characterized by
  \[
  \Theta(\dim U) \geq 0 \quad \forall U \subseteq M.
  \]
- The \( \Theta \)-ss \( \Lambda \)-modules form an abelian subcategory of \( \Lambda \)-mod with Jordan-Hölder series.

- The \( \Theta \)-ss \( \Lambda \)-modules have a coarse moduli space, classifying them up to \( S \)-equivalence. [S for Seshadri]

\( M \sim M' \) \( \iff \) \( M, M' \) have the same simple composition factors in the category of \( \Theta \)-ss modules.
What are the simple objects in the category of $\Theta$-ss modules?

**Theorem. Part 2.**

**Answer:** The $\Theta$-ss modules satisfying: $\Theta(u) = 0 \Rightarrow u \in \{0, M\}$

\[ \forall u \in \{0, M\} \]

Call them the $\Theta$-stable modules.

* If $d$ is indivisible, the $\Theta$-stable modules have a fine moduli space classifying them up to $\cong$.

**Remark.** "All this" generalizes to the non-hereditary case $A = \text{KQ}/\mathcal{I}$. It's just hard to make use of it in general.
A good choice of $\theta$ for local modules if $\mathcal{Q}$ is acyclic

$\mathcal{Q}$ acyclic, $T = S$, simple, $\mathcal{V} = \mathcal{Q}A$. Use module with top...

Define $\theta: Q_0 \to \mathbb{Z}$ via

$\theta(e_i) = 1$ for $i \geq 2$ and

$\theta(e_1) = -\sum_{i=2}^{n} d_i$. Then the local $\mathcal{V}$-modules w. top $T$ and dim vector $d$ are $\theta$-stable.

Corollary: These local modules have a fine moduli space, classifying them up to iso.
PLUSES & MINUSES OF APPROACH(I)

+ :  • Existence of coarse/fine moduli spaces guaranteed by GIT.
    • Since method was very effective for vector bundles of curves, a large arsenal of methods for analyzing the resulting moduli spaces has been established.
    • Semi-invariants of Rep(N, φ) very interesting in their own right.

- :  • Finding "good" Θ's difficult; set of semistable points may be ∅.
    • Θ-(semi-)stability may be hard to interpret in structural terms;
      5-equivalence hard to pin down representation-theoretically.
ADDRESSING THESE PROBLEMS

\[ \Lambda = \mathbb{K} \mathcal{O} \quad \text{(most accessible case)} \]
\[ \Theta : \mathcal{O} \to \mathbb{Z} \quad \text{given.} \]

Known [King, Reineke, Schofield, Harder, Narasimhan]

- \( M \) \( \Theta \)-stable \( \Rightarrow \) \( \text{End}_\Lambda (M) = \mathbb{K} \)

- But: Given \( d \), there is in general no choice of \( \Theta \) such that all \( M \) with \( \dim M = d \) and \( \text{End}_\Lambda (M) = \mathbb{K} \) are \( \Theta \)-stable.

- There is a recursive procedure for deciding whether there are \( \Theta \)-semistable \( \Lambda \)-modules with \( \dim \) vector \( d \).
Slicing \( \text{GRASS}(n, d) \) in Terms of TOPS

**Idea:** Use the benefits of unipotent group actions to better effect by restricting the focus to modules \( M \) with fixed top \( T = M/JM \).

**Intuitive reason for expected gain**

We want to understand when the \( \text{Aut}_n(P) \)-orbits are relatively closed in suitable subvarieties of \( \text{GRASS}(n, d) \).

\[ \text{Aut}_n(P) \cong U \times \text{Aut}_n(P/JP), \]

and the action of \( U \) is easier to analyze than that of \( \text{Aut}_n(P/JP) \). Hence we want to make the latter group as small as possible by cutting down on \( P \), at the expense of obtaining fewer modules as factor modules.
**HERE 'SMALL' IS 'BETTER'**

**THE RESTRICTED MODULE GRASSMANNIAN**

Given: \( d = (d_1, \ldots, d_n) \), \( T \in \Lambda\text{-mod} \) semisimple, \( P \) projective cover of \( T \).

Set \( d' = \dim P - |d| \), and consider \( \text{Grass}_T^d := \{ c \in \text{Gr}(d', P) \mid \lambda C \subseteq \lambda P, \dim \frac{P}{C} = d \} \) with the canonical action of \( \text{Aut}_\lambda (P) \).

Then the \( \text{Aut}_\lambda (P) \)-orbits are in 1-1 correspondence with the isomorphism classes of \( \Lambda\text{-modules} \) \( M \) with \( \dim M = d \) and \( M/\lambda M \cong T \).

The acting group: \( \text{Aut}_\lambda (P) \cong U \times \text{Aut}_\lambda (T) \) with \( U = \{ \text{id}_P + f \mid f \in \text{Hom}_\lambda (P, \lambda P) \} \).
let \( M = \mathbb{P}/c \) with \( c \in \text{Grass}_d^T \),
and suppose \( J^{d+1} = 0 \). Then
\[
S(M) = \left( \frac{M}{J^i M}, \frac{J M}{J^2 M}, \ldots, \frac{J^d M}{J^d M} \right)
\]
is a sequence of semisimple modules with \( M/J M = T \).

For any sequence \( S = (S_0, \ldots, S_L) \) of semisimple modules w. \( S_0 = T \)
and \( \dim S = d \), define
\[
\text{Grass}(S) = \{ c \in \text{Grass}_d^T \mid S(\mathbb{P}/c) = S \}.
\]
Then all \( U \)-orbits are (relatively) closed in \( \text{Grass}(S) \).
SHOOTING FOR ALL MODULES WITH
TOP $T$ & $\dim d$

Thm [H2] Suppose $T$ is squarefree, $P$ its projective cover. Then TFAE:

- The $d$-dim'l modules with top $T$ have a fine moduli space (up to iso).
- The $d$-dim'l modules with top $T$ have a coarse moduli space (up to iso).
- The $(\dim P - d)$-dimensional submodules of $JP$ are invariant under endomorphisms of $P$.
- $\text{Ograss}_d^T = \bigcup_{\text{dim}=d} \text{Ograss}_d$ is the fine moduli space for the $d$-dim'l modules with top $T$.

Cor. [cf King] $T = \Lambda E / J E$ simple, $E J P \subseteq \text{soc } P \Rightarrow$ the equivalent conditions above are satisfied.
Example

\[ \begin{array}{ccc}
& 1 & \\
3 & \xrightarrow{3} & 2 & \xrightarrow{1} & 3 \\
& \downarrow & & \downarrow \\
& \beta & & \beta \\
\end{array} \]

\[ \Lambda = \mathbb{K} \Lambda / \langle \text{paths of length 4} \rangle, \]
\[ T = S_1, \quad \lambda = (2, 3, 2). \]

The fine moduli space for the \( \Lambda \)-modules with top \( T \) and dim vector \( \lambda \) is \( \text{Grass}^T_{\lambda} \approx \text{Flag} (\mathbb{K}^4). \)

Visualization:

Generic \( M \in \text{Grass}^T_{\lambda} \)
SHOOTING FOR THE GRADED MODULES
WITH TOP $T$ & dim $d$

joint with Babson and Thomas

$\Lambda = kQ/I$, $I \subseteq kQ$ homogeneous

Thm. Here "graded" includes "generated in deg 0".

(a) For any simple $T$ and $d \in \mathbb{N}$,
the $d$-dim' graded modules with top $T$ have a fine moduli
space (classifying up to graded iso), namely

$$\bigcup_{d \mid d = \ell} \text{Gr} - \text{Grass}^T_d$$

(b) So do direct sums of local graded modules w. fixed dimensions of the
local summands.

THAT'S AS FAR AS IT GOES.

(c) $T$ arbitrary. If the $d$-dim' graded modules w. top $T$ have
a coarse moduli space, then they are $\oplus$ of local modules.
THREE EASY PIECES
(2 of them easier than Jack Nicholson's)

1. $J^2 = 0$, $T$ simple, $d \in \mathbb{N}$.

Then the irreducible components of the fine moduli space

$$\text{Gr} - \text{Grass}_d^T = \text{Grass}_d^T$$

are direct products of classical Grassmann varieties $\text{Gr}(m; _1 K^n)$.

Idea: Say $T = \Lambda e_1 / je_1$.

Then $P = \Lambda e_1$ looks like this

```
1
1 2 3 ...
```

and the $\Lambda$-submodules of $J^P$ coincide with the $K$-subspaces.
\[ Q = 1 \begin{array}{c}\rightarrow \beta_1 \end{array} 2, \Lambda = \langle Q/\langle\text{paths of length 3}\rangle \rangle \]

\[ T = S_1 \oplus S_2, \quad d = (2, 2) \]

Not all graded modules w. top \( T \) and \( \dim d \) are \( \oplus (\text{graded local}) \), but the graded modules with radical layering \( S = (S_1 \oplus S_2, S_2, S_1) \) are.

Thus they have a fine moduli space \( \text{Gr-OGress}(S) \equiv \mathbb{P}^1 \), corresponding to their normal forms

\[ \begin{array}{c}\frac{1}{2} \end{array} \begin{array}{c}\oplus \end{array} \begin{array}{c}\frac{1}{2} \end{array} \]

\[ A(\cdot \beta_2, \cdot) \]

By contrast, \( \text{OGress}(S) \equiv \mathbb{P}^1 \times \mathbb{P}^1 \) is not a moduli space for the ungraded modules.
3 Not quite so easy:

\[ Q = \bigoplus_{l \geq 3} \mathbb{P}^{(2^l)} \otimes \mathcal{O}_{X^n} \]

\[ \Lambda = \mathcal{U}_Q/\langle \text{all paths of length } L \rangle \]

\( T \) the simple \( \Lambda \)-module, \( \alpha \in \mathbb{N} \).

Then the irreducible components of the fine moduli space

\[ \text{Gr-Op ass} \overline{T} \]

for the graded \( \alpha \)-dim\'l modules with top \( T \) are smooth and rational.
SHOOTING FOR THE MODULES WITHOUT PROPER TOP-STABLE DEGENERATIONS

joint with Derksen and Weyman

**What are they?**

They are the $M \in \mathcal{A}$-mod satisfying:

\[ M \leq M', \ M \neq M' \Rightarrow \text{top}(M) \subsetneq \text{top}(M') \]

\[ \text{degen} \]

Here $M \leq M'$ means: $M'$ is a degener

degeneration of $M$, i.e.

\[ \text{orbit}(M') \subseteq \text{orbit}(M) \]
Theorem. \( T \in \Lambda\)-mod semisimple (arbitrary)

- The modules with dimension vector \( d \) which are degeneration-maximal among those with top \( T \) have a fine moduli space, classifying them up to \( \cong \).

This space, \( \text{Moduli Max}_d^T \), is a projective variety \( \cong \text{Grass}^T_d \).

Satellite result [DHW/Hille]

Any projective variety occurs as \( \text{Moduli Max}_d^T \) for suitable \( \Lambda, T, d \) (even for simple \( T \)).
ANOTHER FACT & A PROBLEM

Fact: \( M \) with \( M/gM = T \) fixed.

The maximal top-stable degenerations of \( M \) have a fine moduli space, classifying up to \( \cong \).

This space is a closed subvariety

\[
\text{Moduli Max}_d^T(M) \subseteq \text{Moduli Max}_d^T.
\]

Question: Which projective varieties occur as \( \text{Moduli Max}_d^T(M) \) for fixed \( M \)?
Theorem (Normal forms for the modules at stake)

For $M \in \Lambda$-mod $T \neq AE$:

(1) $M$ has no proper top-stable degeneration.

(2) $M = \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{t_i} \Lambda e_i / C_{ij}$ for suitable $t_i > 0$ and $C_{ij} \subseteq \Lambda e_i$

such that $C_{i1} \subseteq C_{i2} \subseteq \ldots \subseteq C_{it_i}$, and:

(iii) If $P$ is a projective cover of $M$, then $\dim \text{Hom}_\Lambda (P, JM) = \dim \text{Hom}_\Lambda (M, JM)$

means that first syzygy of $M$ in $P$ is invariant under all homomorphisms $P \rightarrow JP$
Example: We determine the irreducible components of $\text{Moduli}_\Omega^T (M)$ for a specific module $M$.

\[ \lambda = KQ/I, \ I \text{ generated by paths as visible from } \Lambda e_1. \ T = 5. \]

\[ \Lambda e_1 \]

\[ \Lambda e_1/\mathcal{C} \]

\[ M: \]

\[ z_1 \]

\[ z_2 \]

\[ \alpha \beta \]

\[ \omega_1 \omega_2 \omega_3 \omega_4 \omega_5 \omega_6 \]

\[ 1 \]

\[ 2 \]

\[ 1 \]
4 irreducible components $\mathcal{C}_1, \ldots, \mathcal{C}_4$. Generically, their modules look like this:

$\mathcal{C}_1$:

$\begin{align*}
\alpha & \quad \beta \\
2 & \quad 2
\end{align*}$

$\begin{align*}
\alpha & \quad \alpha \\
2 & \quad 2
\end{align*}$

$\begin{align*}
\alpha & \quad \beta \\
2 & \quad 2
\end{align*}$

$\begin{align*}
\omega_1 & \quad \omega_2 \\
1 & \quad 1
\end{align*}$

$\begin{align*}
\omega_3 & \quad \omega_4 \\
1 & \quad 1
\end{align*}$

$\begin{align*}
\omega_5 & \quad \omega_6 \\
1 & \quad 1
\end{align*}$

$\begin{align*}
1 & \quad 1
\end{align*}$

$\begin{align*}
1 & \quad 1
\end{align*}$

$\begin{align*}
1 & \quad 1
\end{align*}$

$\begin{align*}
[k : k : k] & \in \mathbb{P}^2
\end{align*}$

$\begin{align*}
[l : l : l] & \in \mathbb{P}^2
\end{align*}$

$\mathcal{C}_2$:

$\begin{align*}
\alpha & \quad \beta \\
2 & \quad 2
\end{align*}$

$\begin{align*}
\alpha & \quad \alpha \\
2 & \quad 2
\end{align*}$

$\begin{align*}
\alpha & \quad \beta \\
2 & \quad 2
\end{align*}$

$\begin{align*}
\omega_1 & \quad \omega_2 \\
1 & \quad 1
\end{align*}$

$\begin{align*}
\omega_3 & \quad \omega_4 \\
1 & \quad 1
\end{align*}$

$\begin{align*}
\omega_5 & \quad \omega_6 \\
1 & \quad 1
\end{align*}$

$\begin{align*}
1 & \quad 1
\end{align*}$

$\begin{align*}
1 & \quad 1
\end{align*}$

$\begin{align*}
1 & \quad 1
\end{align*}$

$\begin{align*}
[k : k : k] & \in \mathbb{P}^2
\end{align*}$

$\mathcal{C}_3$:

$\begin{align*}
\alpha & \quad \beta \\
2 & \quad 2
\end{align*}$

$\begin{align*}
\alpha & \quad \alpha \\
2 & \quad 2
\end{align*}$

$\begin{align*}
\alpha & \quad \beta \\
2 & \quad 2
\end{align*}$

$\begin{align*}
\omega_1 & \quad \omega_2 \\
1 & \quad 1
\end{align*}$

$\begin{align*}
\omega_3 & \quad \omega_4 \\
1 & \quad 1
\end{align*}$

$\begin{align*}
\omega_5 & \quad \omega_6 \\
1 & \quad 1
\end{align*}$

$\begin{align*}
1 & \quad 1
\end{align*}$

$\begin{align*}
1 & \quad 1
\end{align*}$

$\begin{align*}
1 & \quad 1
\end{align*}$

$\begin{align*}
[l : l : l'] & \in \mathbb{P}^2
\end{align*}$

$\mathcal{C}_4$:

$\begin{align*}
\alpha & \quad \beta \\
2 & \quad 2
\end{align*}$

$\begin{align*}
\alpha & \quad \alpha \\
2 & \quad 2
\end{align*}$

$\begin{align*}
\alpha & \quad \beta \\
2 & \quad 2
\end{align*}$

$\begin{align*}
\omega_1 & \quad \omega_2 \\
1 & \quad 1
\end{align*}$

$\begin{align*}
\omega_3 & \quad \omega_4 \\
1 & \quad 1
\end{align*}$

$\begin{align*}
\omega_5 & \quad \omega_6 \\
1 & \quad 1
\end{align*}$

$\begin{align*}
1 & \quad 1
\end{align*}$

$\begin{align*}
1 & \quad 1
\end{align*}$

$\begin{align*}
1 & \quad 1
\end{align*}$

$\begin{align*}
1 & \quad 1
\end{align*}$

$\begin{align*}
\omega_2 / \omega_6 & \\
1 & \quad 1
\end{align*}$

$\begin{align*}
\omega_3 & \quad \omega_4 \\
1 & \quad 1
\end{align*}$

$\begin{align*}
\omega_5 & \quad \omega_6 \\
1 & \quad 1
\end{align*}$

$\begin{align*}
1 & \quad 1
\end{align*}$

$\begin{align*}
1 & \quad 1
\end{align*}$

$\begin{align*}
1 & \quad 1
\end{align*}$

$\mathcal{C}_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{C}_2 \cong \mathcal{C}_3 \cong \mathbb{P}^1$, $\mathcal{C}_4 \cong \mathbb{P}^0$
4 irreducible components $\mathcal{C}_1, \ldots, \mathcal{C}_4$. Generically, their modules look like this:

$\mathcal{C}_1$: 

$\mathcal{C}_2$: 

$\mathcal{C}_3$: 

$\mathcal{C}_4$: 

$\mathcal{C}_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{C}_2 \cong \mathcal{C}_3 \cong \mathbb{P}^1$, $\mathcal{C}_4 \cong \mathbb{P}^0$
PLUSES & MINUSES OF APPROACH

+: We control the class of modules we are trying to classify (defined in representation-theoretic terms) as well as the equivalence relation, up to which we are classifying (usually $\cong$, or iso preserving additional structure).

-: In the above instances, the moduli spaces are "largely" computable and analyzable in terms of combinatorics of quiver & reln's.

-: Large classes of rep-theoretically defined modules "rarely" have moduli spaces. Existence proofs rely on ad-hoc methods, depending on the class of modules considered.

-: No "ready-made" arsenal available for analysis of resulting moduli spaces, existence provided.