Chapter 4

4.1 (a) We are given \( p = 2 \), \( \mu = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \), \( \Sigma = \begin{bmatrix} \frac{1}{2} & -0.8 \times \sqrt{2} \\ -0.8 \times \sqrt{2} & 1 \end{bmatrix} \) so \( |\Sigma| = 0.72 \) and

\[
\Sigma^{-1} = \begin{bmatrix} \frac{1}{0.72} & \sqrt{2} \\ \sqrt{2} & \frac{2}{0.72} \end{bmatrix}
\]

\[
f(x) = \frac{1}{(2\pi)^{1/2}} \exp \left( -\frac{1}{2} \left[ \frac{1}{0.72} (x_1 - 1)^2 + \frac{2\sqrt{2}}{0.9} (x_1 - 1)(x_2 - 3) + \frac{2}{0.72} (x_2 - 3)^2 \right] \right)
\]

(b)

\[
\frac{1}{0.72} (x_1 - 1)^2 + \frac{2\sqrt{2}}{0.9} (x_1 - 1)(x_2 - 3) + \frac{2}{0.72} (x_2 - 3)^2
\]

4.2 (a) We are given \( p = 2 \), \( \mu = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \), \( \Sigma = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 \end{bmatrix} \) so \( |\Sigma| = 3/2 \) and

\[
\Sigma^{-1} = \begin{bmatrix} \frac{2}{\sqrt{3}} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{2}{\sqrt{3}} \end{bmatrix}
\]

\[
f(x) = \frac{1}{(2\pi)^{1/2}} \exp \left( -\frac{1}{2} \left[ \frac{2}{3} x_1^2 - \frac{2\sqrt{2}}{3} x_1(x_2 - 2) + \frac{4}{3} (x_2 - 2)^2 \right] \right)
\]

(b)

\[
\frac{2}{3} x_1^2 - \frac{2\sqrt{2}}{3} x_1(x_2 - 2) + \frac{4}{3} (x_2 - 2)^2
\]

(c) \( c^2 = \chi^2(2)(0.5) = 1.39 \). Ellipse centered at \([0,2]^t\) with the major axis having half-length \( \sqrt{\lambda_1} c = \sqrt{2.366}\sqrt{1.39} = 1.81 \). The major axis lies in the direction \( e = [0.888, 0.460]^t \). The minor axis lies in the direction \( e = [-0.460, 0.888]^t \) and has half-length \( \sqrt{\lambda_2} c = \sqrt{0.634}\sqrt{1.39} = .94 \).
4.3 We apply Result 4.5 that relates zero covariance to statistical independence

a) No, $\sigma_{12} \neq 0$

b) Yes, $\sigma_{23} = 0$

c) Yes, $\sigma_{13} = \sigma_{23} = 0$

d) Yes, by Result 4.3, $(X_1+X_2)/2$ and $X_3$ are jointly normal and their covariance is $\frac{1}{2}\sigma_{13} + \frac{1}{2}\sigma_{23} = 0$.

e) No, by Result 4.3 with $A = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{5}{2} & 1 & -1 \end{bmatrix}$, form $A \neq A'$ to see that the covariance is 10 and not 0.
4.4  
   a) \(3X_1 - 2X_2 + X_3\) is \(N(13, 9)\)
   
   b) Require \(\text{Cov}(X_2, X_2 - a_1 X_1 - a_3 X_3) = 3 - a_1 - 2a_3 = 0\). Thus any \(a' = [a_1, a_3]\) of the form \(a' = [3 - 2a_3, a_3]\) will meet the requirement. As an example, \(a' = [1, 1]\).

4.5  
   a) \(X_1 | x_2\) is \(N(\frac{1}{\sqrt{2}} (x_2 - 2), \frac{3}{2})\)
   
   b) \(X_2 | x_1, x_3\) is \(N(-2x_1 - 5, 1)\)
   
   c) \(X_3 | x_1, x_2\) is \(N(\frac{1}{2} (x_1 + x_2 + 3), \frac{1}{2})\)

4.6  
   (a) \(X_1\) and \(X_2\) are independent since they have a bivariate normal distribution with covariance \(\sigma_{12} = 0\).
   
   (b) \(X_1\) and \(X_3\) are dependent since they have nonzero covariance \(\sigma_{13} = -1\).
   
   (c) \(X_2\) and \(X_3\) are independent since they have a bivariate normal distribution with covariance \(\sigma_{23} = 0\).
   
   (d) \(X_1, X_3\) and \(X_2\) are independent since they have a trivariate normal distribution where \(\sigma_{12} = 0\) and \(\sigma_{32} = 0\).
   
   (e) \(X_1\) and \(X_1 + 2X_2 - 3X_3\) are dependent since they have nonzero covariance \(\sigma_{11} + 2\sigma_{12} - 3\sigma_{13} = 4 + 2(0) - 3(-1) = 7\)

4.7  
   (a) \(X_1 | x_3\) is \(N(1 + .5(x_3 - 2), 3.5)\)
   
   (b) \(X_1 | x_2, x_3\) is \(N(1 + .5(x_3 - 2), 3.5)\). Since \(X_2\) is independent of \(X_1\), conditioning further on \(x_2\) does not change the answer from Part a).
4.15 First,
\[
\sum_{j=1}^{n} (x_j - \bar{x})(x_j - \bar{x})' = (x - \bar{x})' \left( \sum_{j=1}^{n} (x_j - \bar{x}) \right) = (x - \bar{x})(n\bar{x} - n\bar{x})' = 0
\]
Also,
\[
\sum_{j=1}^{n} (x_j - \bar{x})(x_j - \bar{x})' = \left( \sum_{j=1}^{n} (x_j - \bar{x})(x_j - \bar{x})' \right)' = 0'
\]

4.16 (a) By Result 4.8, with \( c_1 = c_2 = c_3 = c_4 = \frac{1}{4} \), \( c_5 = c_6 = -\frac{1}{4} \) and \( \mu_j = \mu \) for \( j = 1, ..., 4 \) we have \( \sum_{j=1}^{4} c_j \mu_j = 0 \) and \( ( \sum_{j=1}^{4} c_j^2 ) \Sigma = \frac{1}{4} \Sigma \). Consequently, \( V_1 \) is \( N(0, \frac{1}{4} \Sigma ) \). Similarly, setting \( b_1 = b_2 = 1/4 \) and \( b_3 = b_4 = -1/4 \), we find that \( V_2 \) is \( N(0, \frac{1}{4} \Sigma ) \).

(b) Again by Result 4.8, we know that \( V_1 \) and \( V_2 \) are jointly multivariate normal with covariance
\[
\Sigma = \left( \begin{array}{cccc}
\frac{1}{4} & \frac{-1}{4} & \frac{1}{4} & \frac{-1}{4} \\
\frac{-1}{4} & \frac{1}{4} & \frac{-1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{-1}{4} & \frac{1}{4} & \frac{-1}{4} \\
\frac{-1}{4} & \frac{1}{4} & \frac{-1}{4} & \frac{1}{4}
\end{array} \right) = 0
\]
That is,
\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
\]
so the joint density of the 2p variables is
\[
f(v_1, v_2) = \frac{1}{(2\pi)^p |\frac{1}{4} \Sigma|} \exp \left( -\frac{1}{2} \begin{bmatrix} v_1', v_2' \end{bmatrix} \left( \begin{bmatrix} \frac{1}{4} \Sigma & 0 \\
0 & \frac{1}{4} \Sigma \end{bmatrix} \right)^{-1} \begin{bmatrix} v_1 \end{bmatrix} \right)
\]
\[
= \frac{1}{(2\pi)^p |\frac{1}{4} \Sigma|} \exp \left( -\frac{1}{2} \begin{bmatrix} v_1' \Sigma^{-1} v_1 + v_2' \Sigma^{-1} v_2 \end{bmatrix} \right)
\]

4.17 By Result 4.8, with \( c_1 = c_2 = c_3 = c_4 = c_5 = 1/5 \) and \( \mu_j = \mu \) for \( j = 1, ..., 5 \) we find that \( V_1 \) has mean \( \sum_{j=1}^{5} c_j \mu_j = \mu \) and covariance matrix \( ( \sum_{j=1}^{5} c_j^2 ) \Sigma = \frac{1}{5} \Sigma \).

Similarly, setting \( b_1 = b_3 = b_5 = 1/5 \) and \( b_2 = b_4 = -1/5 \) we find that \( V_2 \) has mean \( \sum_{j=1}^{5} b_j \mu_j = \frac{1}{5} \mu \) and covariance matrix \( ( \sum_{j=1}^{5} b_j^2 ) \Sigma = \frac{1}{5} \Sigma \).

Again by Result 4.8, we know that \( V_1 \) and \( V_2 \) have covariance
\[
( \sum_{i=1}^{4} b_j c_j ) \Sigma = \left( \begin{array}{cccc}
\frac{1}{5} & \frac{-1}{5} & \frac{1}{5} & \frac{-1}{5} \\
\frac{-1}{5} & \frac{1}{5} & \frac{-1}{5} & \frac{1}{5} \\
\frac{1}{5} & \frac{-1}{5} & \frac{1}{5} & \frac{-1}{5} \\
\frac{-1}{5} & \frac{1}{5} & \frac{-1}{5} & \frac{1}{5}
\end{array} \right) \Sigma = \frac{1}{25} \Sigma
\]
4.18  By Result 4.11 we know that the maximum likelihood estimates of $\mu$ and $\Sigma$ are $\bar{x} = [4,6]'$ and


= $\frac{1}{4} \left[ \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{array} \right] + \frac{1}{4} \left[ \begin{array}{ccc} 0 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \left( \begin{array}{ccc} 0 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)$

= $\frac{1}{4} \left[ \begin{array}{ccc} 2 & 1 \\ 1 & 6 \end{array} \right]$

4.19  a)  By Result 4.7 we know that $(x_1 - \mu)' \Sigma^{-1} (x_1 - \mu) \sim \chi^2_6$

b)  From (4-23), $\bar{x} \sim N_6(\mu, \frac{1}{20} \Sigma)$. Then $\bar{x} - \mu \sim N_6(0, \frac{1}{20} \Sigma)$ and finally $\sqrt{20} (\bar{x} - \mu) \sim N_6(0, \Sigma)$

c)  From (4-23), $\chi_9$ has a Wishart distribution with 19 d.f.

4.20  $B(\chi_9)B'$ is a 2x2 matrix distributed as $W_{19}(-B\Sigma B')$ with 19 d.f. where

a)  $B\Sigma B'$ has

$(1,1)$ entry = $\sigma_{11} + \frac{1}{4}\sigma_{22} + \frac{1}{4}\sigma_{33} - \sigma_{12} - \sigma_{13} + \frac{1}{2}\sigma_{23}$

$(1,2)$ entry = $\frac{1}{2}\sigma_{14} + \frac{1}{4}\sigma_{24} + \frac{1}{4}\sigma_{34} - \frac{1}{2}\sigma_{15} + \frac{1}{4}\sigma_{25} + \frac{1}{4}\sigma_{35} + \sigma_{16} - \frac{1}{2}\sigma_{26} - \frac{1}{2}\sigma_{36}$

$(2,2)$ entry = $\sigma_{66} + \frac{1}{4}\sigma_{55} + \frac{1}{4}\sigma_{44} - \sigma_{45} - \sigma_{56} + \frac{1}{2}\sigma_{45}$

b)  $B\Sigma B'$ = $\begin{bmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{bmatrix}$
4.21  (a) $\bar{X}$ is distributed $N_4(\mu, n^{-1}\Sigma)$

(b) $X_1 - \mu$ is distributed $N_4(0, \Sigma)$ so $(X_1 - \mu)'\Sigma^{-1}(X_1 - \mu)$ is distributed as chi-square with $p$ degrees of freedom.

(c) Using Part a),

$$(\bar{X} - \mu)'(n^{-1}\Sigma)^{-1}(\bar{X} - \mu) = n(\bar{X} - \mu)'\Sigma^{-1}(\bar{X} - \mu)$$

is distributed as chi-square with $p$ degrees of freedom.

(d) Approximately distributed as chi-square with $p$ degrees of freedom. Since the sample size is large, $\Sigma$ can be replaced by $S$. 
4.22 a) We see that \( n = 75 \) is a sufficiently large sample (compared with \( p \)) and apply Result 4.13 to get \( \sqrt{n}(\bar{X} - \mu) \) is approximately \( N_p(0, \frac{1}{n \mu^2}) \) and that \( \bar{X} \) is approximately \( N_p(\mu, \frac{1}{n \mu^2}) \).

b) By (4-28) we conclude that \( \sqrt{n}(\bar{X} - \mu)'s^{-1}(\bar{X} - \mu) \) is approximately \( X_p^2 \).

4.23 (a) The Q-Q plot shown below is not particularly straight, but the sample size \( n = 10 \) is small. Difficult to determine if data are normally distributed from the plot.

![Q-Q Plot for Dow Jones Data](image)

(b) \( r_Q = .95 \) and \( n = 10 \). Since \( r_Q = .95 > .9351 \) (see Table 4.2), cannot reject hypothesis of normality at the 10% level.
4.24 (a) Q-Q plots for sales and profits are given below. Plots not particularly straight, although Q-Q plot for profits appears to be "straighter" than plot for sales. Difficult to assess normality from plots with such a small sample size ($n = 10$).

(b) The critical point for $n = 10$ when $\alpha = .10$ is .9351. For sales, $r_Q = .940$ and for profits, $r_Q = .968$. Since the values for both of these correlations are greater than .9351, we cannot reject normality in either case.
4.25 The chi-square plot for the world's largest companies data is shown below. The plot is reasonably straight and it would be difficult to reject multivariate normality given the small sample size of \( n = 10 \). Information leading to the construction of this plot is also displayed.

\[
\bar{x} = \begin{bmatrix} 155.6 \\ 14.7 \\ 710.9 \end{bmatrix} \quad S = \begin{bmatrix} 7476.5 & 303.6 & -35576 \\ 303.6 & 26.2 & -1053.8 \\ -35576 & -1053.8 & 237054 \end{bmatrix}
\]

<table>
<thead>
<tr>
<th>Ordered SqDist</th>
<th>Chi-square Quantiles</th>
</tr>
</thead>
<tbody>
<tr>
<td>.3142</td>
<td>.3518</td>
</tr>
<tr>
<td>1.2894</td>
<td>.7978</td>
</tr>
<tr>
<td>1.4073</td>
<td>1.2125</td>
</tr>
<tr>
<td>1.6418</td>
<td>1.6416</td>
</tr>
<tr>
<td>2.0195</td>
<td>2.1095</td>
</tr>
<tr>
<td>3.0411</td>
<td>2.6430</td>
</tr>
<tr>
<td>3.1891</td>
<td>3.2831</td>
</tr>
<tr>
<td>4.3520</td>
<td>4.1083</td>
</tr>
<tr>
<td>4.8365</td>
<td>5.3170</td>
</tr>
<tr>
<td>4.9091</td>
<td>7.8147</td>
</tr>
</tbody>
</table>
4.26 (a) \( \bar{x} = \begin{bmatrix} 5.20 \\ 12.48 \end{bmatrix} \), \( S = \begin{bmatrix} 10.6222 & -17.7102 \\ -17.7102 & 30.8544 \end{bmatrix} \), \( S^{-1} = \begin{bmatrix} 2.1898 & 1.2569 \\ 1.2569 & .7539 \end{bmatrix} \)

Thus \( d_j^2 = 1.8753, 2.0203, 2.9009, .7353, .3105, .0176, 3.7329, .8165, 1.3753, 4.2153 \)

(b) Since \( \chi^2_2(.5) = 1.39 \), 5 observations (50%) are within the 50% contour.

(c) The chi-square plot is shown below.

(d) Given the results in parts (b) and (c) and the small number of observations \( n = 10 \), it is difficult to reject bivariate normality.
The Q-Q plot is reasonably straight. $r_Q = .978$ ($\lambda = 0$)

For $\lambda = 1/4$, $r_Q = .993$ so $\lambda = 1/4$ is a little better choice for the normalizing transformation.

Since $r_Q = .970 < .973$ (See Table 4.2 for $n = 40$ and $\alpha = .05$), we would reject the hypothesis of normality at the 5% level.
4.29

(a).

\[ \bar{x} = \left( \begin{array}{c} 10.046719 \\ 9.4047619 \end{array} \right), \quad S = \left( \begin{array}{cc} 11.363531 & 3.126597 \\ 30.978513 & \end{array} \right) \]

Generalized distances are as follows:

\[
\begin{array}{cccccc}
0.4607 & 0.6592 & 2.3771 & 1.6283 & 0.4135 & 0.4761 & 1.1849 \\
10.6392 & 0.1388 & 0.8162 & 1.3566 & 0.6228 & 5.6494 & 0.3159 \\
0.4135 & 0.1225 & 0.8988 & 4.7647 & 3.0089 & 0.6592 & 2.7741 \\
1.0360 & 0.7874 & 3.4438 & 6.1489 & 1.0360 & 0.1388 & 0.8856 \\
0.1380 & 2.2489 & 0.1901 & 0.4607 & 1.1472 & 7.0857 & 1.4584 \\
0.1225 & 1.8985 & 2.7783 & 8.4731 & 0.6370 & 0.7032 & 1.8014 \\
\end{array}
\]

(b). The number of observations whose generalized distances are less than \( \chi^2(0.5) = 1.39 \) is 26. So the proportion is \( 26/42 = 0.6190 \).

(c). CHI-SQUARE PLOT FOR \((X_1 \ X_2)\)

4.30 (a) \( \hat{\lambda}_1 = 0.5 \) but \( \hat{\lambda}_1 = 1 \) (i.e. no transformation) not ruled out by data. For \( \hat{\lambda}_1 = 1, \quad r_2 = .981 > .9351 \) the critical point for testing normality with \( n = 10 \) and \( \alpha = .10 \). We cannot reject the hypothesis of normality at the 10% level (and, consequently, not at the 5% level).

(b) \( \hat{\lambda}_1 = 1 \) (i.e. no transformation). For \( \hat{\lambda}_1 = 1, \quad r_2 = .971 > .9351 \) the critical point for testing normality with \( n = 10 \) and \( \alpha = .10 \). We cannot reject the hypothesis of normality at the 10% level (and, consequently, not at the 5% level).

(c) The likelihood function \( l(\hat{\lambda}_1, \hat{\lambda}_2) \) is fairly flat in the region of \( \hat{\lambda}_1 = 1, \hat{\lambda}_2 = 1 \) so these values are not ruled out by the data. These results are consistent with those in parts (a) and (b).

\( Q-Q \) plots follow.
The non-multiple-sclerosis group:

<table>
<thead>
<tr>
<th></th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_Q$</td>
<td>0.94482*</td>
<td>0.96133*</td>
<td>0.95585*</td>
<td>0.97574*</td>
<td>0.94446*</td>
</tr>
<tr>
<td>Transformation</td>
<td>$X_1^{0.5}$</td>
<td>$X_2^{-3.5}$</td>
<td>$(X_3 + 0.005)^{0.4}$</td>
<td>$X_4^{-3.4}$</td>
<td>$(X_5 + 0.005)^{0.32}$</td>
</tr>
</tbody>
</table>

*: significant at 5 % level (the critical point = 0.9826 for n=69).

The multiple-sclerosis group:

<table>
<thead>
<tr>
<th></th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_Q$</td>
<td>0.97137</td>
<td>0.97209</td>
<td>0.79523*</td>
<td>0.97869</td>
<td>0.84135*</td>
</tr>
<tr>
<td>Transformation</td>
<td>$X_1^{0.5}$</td>
<td>$X_2^{-3.5}$</td>
<td>$(X_3 + 0.005)^{0.26}$</td>
<td>$X_4^{-3.4}$</td>
<td>$(X_5 + 0.005)^{0.21}$</td>
</tr>
</tbody>
</table>

*: significant at 5 % level (the critical point = 0.9640 for n=29).

Transformations of $X_3$ and $X_4$ do not improve the approximation to normality very much because there are too many zeros.

4.32

<table>
<thead>
<tr>
<th></th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_5$</th>
<th>$X_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_Q$</td>
<td>0.98464*</td>
<td>0.94526*</td>
<td>0.9970</td>
<td>0.98098*</td>
<td>0.99057</td>
<td>0.92779*</td>
</tr>
<tr>
<td>Transformation</td>
<td>$(X_1 + 0.005)^{-0.59}$</td>
<td>$X_2^{-0.49}$</td>
<td>$X_3^{0.25}$</td>
<td>$X_4^{0.25}$</td>
<td>$(X_5 + 0.005)^{0.51}$</td>
<td></td>
</tr>
</tbody>
</table>

*: significant at 5 % level (the critical point = 0.9870 for n=98).

4.33

Marginal Normality:

<table>
<thead>
<tr>
<th></th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_Q$</td>
<td>0.95986*</td>
<td>0.95039*</td>
<td>0.96341</td>
<td>0.98079</td>
</tr>
</tbody>
</table>

*: significant at 5 % level (the critical point = 0.9652 for n=30).

Bivariate Normality: the $\chi^2$ plots are given in the next page. Those for $(X_1, X_2), (X_1, X_3), (X_3, X_4)$ appear reasonably straight.
4.34
Marginal Normality:

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_5$</th>
<th>$X_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_Q$</td>
<td>0.95162*</td>
<td>0.97209</td>
<td>0.98421</td>
<td>0.99011</td>
<td>0.98124</td>
</tr>
</tbody>
</table>

*: significant at 5 % level (the critical point = 0.9591 for n=25).

Bivariate Normality: Omitted.

4.35 Marginal normality:

\[
\begin{align*}
X_1 \text{ (Density)} & \quad X_2 \text{ (MachDir)} & \quad X_3 \text{ (CrossDir)} \\
\rho & \quad .897* & \quad .991 & \quad .924*
\end{align*}
\]

* significant at the 5% level; critical point = .974 for n = 41

From the chi-square plot (see below), it is obvious that observation #25 is a multivariate outlier. If this observation is removed, the chi-square plot is considerably more "straight line like" and it is difficult to reject a hypothesis of multivariate normality. Moreover, $r_Q$ increases to .979 for density, it is virtually unchanged (.992) for machine direction and cross direction (.926).
4.36 Marginal normality:

<table>
<thead>
<tr>
<th></th>
<th>100m</th>
<th>200m</th>
<th>400m</th>
<th>800m</th>
<th>1500m</th>
<th>3000m</th>
<th>Marathon</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_Q )</td>
<td>.983</td>
<td>.976*</td>
<td>.969*</td>
<td>.952*</td>
<td>.909*</td>
<td>.866*</td>
<td>.859*</td>
</tr>
</tbody>
</table>

* significant at the 5% level; critical point = .978 for \( n = 54 \)

Notice how the values of \( r_Q \) decrease with increasing distance. As the distance increases, the distribution of times becomes increasingly skewed to the right.

The chi-square plot is not consistent with multivariate normality. There are several multivariate outliers.

4.37 Marginal normality:

<table>
<thead>
<tr>
<th></th>
<th>100m</th>
<th>200m</th>
<th>400m</th>
<th>800m</th>
<th>1500m</th>
<th>3000m</th>
<th>Marathon</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_Q )</td>
<td>.989</td>
<td>.985</td>
<td>.984</td>
<td>.968*</td>
<td>.947*</td>
<td>.929*</td>
<td>.921*</td>
</tr>
</tbody>
</table>

* significant at the 5% level; critical point = .978 for \( n = 54 \)

As measured by \( r_Q \), times measured in meters/second for the various distances are more nearly marginally normal than times measured in seconds or minutes (see Exercise 4.36). Notice the values of \( r_Q \) decrease with increasing distance. In this case, as the distance increases the distribution of times becomes increasingly skewed to the left.

The chi-square plot is not consistent with multivariate normality. There are several multivariate outliers.
4.38. Marginal and multivariate normality of bull data

Normality of Bull Data

A chi-square plot of the ordered distances

$r = 0.9916$ normal

$r = 0.9631$ not normal

$r = 0.9847$ normal

Quantiles of Standard Normal

$r = 0.9376$ not normal

$r = 0.9956$ normal

$r = 0.9934$ normal
From Table 4.2, with $\alpha = 0.05$ and $n = 76$, the critical point for the $Q - Q$ plot correlation coefficient test for normality is 0.9839. We reject the hypothesis of multivariate normality at $\alpha = 0.05$, because some marginals are not normal.
4.39 (a) Marginal normality:

<table>
<thead>
<tr>
<th></th>
<th>independence</th>
<th>support</th>
<th>benevolence</th>
<th>conformity</th>
<th>leadership</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_Q$</td>
<td>.991</td>
<td>.993</td>
<td>.997</td>
<td>.997</td>
<td>.984*</td>
</tr>
</tbody>
</table>

* significant at the 5% level; critical point = .990 for $n = 130$

(b) The chi-square plot is shown below. Plot is straight with the exception of observation #60. Certainly if this observation is deleted would be hard to argue against multivariate normality.

(c) Using the $r_Q$ statistic, normality is rejected at the 5% level for leadership. If leadership is transformed by taking the square root (i.e. $\lambda = 0.5$), $r_Q = .998$ and we cannot reject normality at the 5% level.
4.40 (a) Scatterplot is shown below. Great Smoky park is an outlier.

![Scatterplot of Size vs Visitors](image)

(b) The power transformation $\lambda_1 = 0.5$ (i.e. square root) makes the size observations more nearly normal. $r_Q = .904$ before transformation and $r_Q = .975$ after transformation. The 5% critical point with $n = 15$ for the hypothesis of normality is .9389. The Q-Q plot for the transformed observations is given below.

![Q-Q Plot for Square Root Size](image)

(c) The power transformation $\lambda_2 = 0$ (i.e. logarithm) makes the visitor observations more nearly normal. $r_Q = .837$ before transformation and $r_Q = .960$ after transformation. The 5% critical point with $n = 15$ for the hypothesis of normality is .9389. The Q-Q plot for the transformed observations is given next.
A chi-square plot for the transformed observations is shown below. Given the small sample size \( n = 15 \), the plot is reasonably straight and it would be hard to reject bivariate normality.
4.41 (a) Scatterplot is shown below. There do not appear to be any outliers with the possible exception of observation #21.

(b) The power transformation $\hat{\lambda} = 0$ (i.e. logarithm) makes the duration observations more nearly normal. $r_Q = .958$ before transformation and $r_Q = .989$ after transformation. The 5% critical point with $n = 25$ for the hypothesis of normality is .9591. The Q-Q plot for the transformed observations is given below.
(c) The power transformation $\lambda_2 = -0.5$ (i.e. reciprocal of square root) makes the man/machine time observations more nearly normal. $r_Q = .939$ before transformation and $r_Q = .991$ after transformation. The 5% critical point with $n = 25$ for the hypothesis of normality is .9591. The $Q-Q$ plot for the transformed observations is given next.

(d) A chi-square plot for the transformed observations is shown below. The plot is straight and it would be difficult to reject bivariate normality.