Vandermonde Matrices

Let

\[ A = \begin{bmatrix}
1 & k_1 & k_1^2 & \cdots & k_1^{n-1} \\
1 & k_2 & k_2^2 & \cdots & k_2^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & k_n & k_n^2 & \cdots & k_n^{n-1}
\end{bmatrix} \]

where \( k_1, k_2, \ldots, k_n \) are any scalars. A matrix of this form is called a Vandermonde matrix, after Alexandre Theophile Vandermonde (1735-1796).

Let us find the determinant of \( A \).

Observe that

\[ AT = \begin{bmatrix}
1 & B \\
1 & 0
\end{bmatrix} \quad (1) \]

where

\[ T = \begin{bmatrix}
1 & -k_n & 0 & \cdots & 0 & 0 \\
0 & 1 & -k_n & 0 & 0 \\
0 & 0 & 1 & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -k_n \\
0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix} \]

and

\[ B = \begin{bmatrix}
k_1-k_n & k_1(k_1-k_n) & \cdots & k_1^{n-2}(k_1-k_n) \\
k_2-k_n & k_2(k_2-k_n) & \cdots & k_2^{n-2}(k_2-k_n) \\
\vdots & \vdots & \ddots & \vdots \\
k_{n-1}-k_n & k_{n-1}(k_{n-1}-k_n) & \cdots & k_{n-1}^{n-2}(k_{n-1}-k_n)
\end{bmatrix} \]
Note that the effect of postmultiplying $A$ by $T$ is to add, to the $j$th column of $A$, a scalar multiple of the preceding column ($j = 2, \ldots, n$), thus creating a matrix whose last row is $(1, 0, 0, \ldots, 0)$.

Observe also that $B$ is expressible as

$$B = DC$$

where

$$D = \text{diag}(k_1 - k_n, k_2 - k_n, \ldots, k_{n-1} - k_n)$$

and

$$C = \begin{bmatrix}
1 & k_1 & k_1^2 & \cdots & k_1^{n-2} \\
1 & k_2 & k_2^2 & \cdots & k_2^{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
1 & k_{n-1} & k_{n-1}^2 & \cdots & k_{n-1}^{n-2}
\end{bmatrix}$$

Thus, $B$ is expressible as a product of a diagonal matrix and of the $(n-1)$ x $(n-1)$ submatrix of $A$ obtained by deleting the last row and column of $A$. Note that this submatrix (i.e., the matrix $C$) is an $(n-1)$ x $(n-1)$ Vandermonde matrix.

Making use of the decompositions (1) and (2) and of basic properties of determinants, we find that

$$|A| = |A||T| = |AT|$$

$$= \begin{vmatrix} 1 & B \\
\phantom{1} & 0 \end{vmatrix} = (-1)^{n-1} \begin{vmatrix} 1 & 0 \\
\phantom{1} & B \end{vmatrix}$$

$$= (-1)^{n-1}|B| = (-1)^{n-1}|D||C|$$

$$= (-1)^{n-1}(k_1 - k_n)(k_2 - k_n) \cdots (k_{n-1} - k_n)|C|$$

$$= (k_n - k_1)(k_n - k_2) \cdots (k_n - k_{n-1})|C|$$

Formula (3) serves to relate the determinant of an $n \times n$ Vandermonde matrix to that of an $(n-1) \times (n-1)$ Vandermonde matrix, and its repeated application allows us to evaluate the determinant of any Vandermonde matrix.
Clearly, when \( n = 2 \),
\[
|A| = k_2 - k_1 ;
\]
when \( n = 3 \)
\[
|A| = (k_3 - k_1)(k_3 - k_2)(k_2 - k_1) ;
\]
and, in general,
\[
\cdots (k_2 - k_1)
\]
as can be formally verified by a simple mathematical induction argument based on the relationship (3).

It is evident from formula (4) that \( |A| \neq 0 \) if and only if \( k_j \neq k_i \) for \( j > 1 = 1, \ldots, n \). Thus, \( A \) is nonsingular if and only if the \( n \) scalars \( k_1, k_2, \ldots, k_n \) are distinct.