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GALERKIN METHODS IN AGE AND SPACE FOR A POPULATION MODEL WITH NONLINEAR DIFFUSION*

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Abstract. We present Galerkin methods in both the age and space variables for an age-dependent population undergoing nonlinear diffusion. The methods presented are a generalization of methods, where the approximation space in age is the space of piecewise constant functions. In this paper, we allow the use of discontinuous piecewise polynomial subspaces of L^2 as the approximation space in age. As in the piecewise constant case, we move the discretization along characteristic lines. The time variable has been left continuous. The methods are shown to be superconvergent in the age variable.

Key words. population dynamics, age-dependence, nonlinear diffusion, Galerkin methods, superconvergence

AMS subject classifications. Primary, 65M60; Secondary, 35Q80, 65M15, 92D25

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1. Introduction.

We present Galerkin methods in both the age and space variables for a model of an age-dependent population undergoing nonlinear diffusion. The methods presented are a generalization of methods presented in [2], where the approximation space in age was taken to be the space of piecewise constant functions. The use, analysis, and numerical solution of models with dependence on age and time, and of models that also include space, is discussed in [2] and references therein.

In this paper, we allow the use of discontinuous piecewise polynomial subspaces of L^2 as the approximation space in age. As in the piecewise constant case, we move the discretization along characteristic lines. This preserves the important fact that age and time advance together and that the resulting discretization will be dispersion-free.

Some previous numerical methods [8, 9, 12] for age-structured models with spatial diffusion also discretized along characteristics, but they did so simultaneously in age and time and thus imposed the often crippling constraint that the time and age steps be both constant and equal. The difficulty with this approach is twofold. First, the use of constant age and time steps prevents adaptivity of the discretization in age or, especially, time. Second, and more importantly, the coupling of the age and time meshes can cause great losses of efficiency since only rarely will the dynamics in time be on the same scale as the dynamics in age. This is particularly the case when space is involved since sharp moving fronts can require small time steps, whereas the behavior in the age variable can remain relatively smooth. A computational example illustrating the advantages of decoupling age and time is presented in [2] and for the context of *Proteus mirabilis* swarm colony development [6, 14] in Chapter 4 of [1].

The age discretization presented in [8, 9, 12] can be viewed as special cases of the methods presented here and in [2] by setting the time and age meshes to be constant

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and equal and using a backward Euler discretization in time and a piecewise constant finite element space in age.

The importance of allowing different age and time discretizations is perhaps illustrated by the application of the methods of de Roos [3]. These methods have found use in the study of ecological systems such as *Daphnia* (see [4] and the references therein) as well as in theoretical population biology [10, 13]. The methods of de Roos are formulated for the case of time and a variable representing some sort of physiological structure, most simply age, and involve moving the age nodes along characteristics. However, the representation of the approximate solution is probabilistic and not functional, and birth and death are handled differently than in this paper. Even so, it would be interesting to know if an energy analysis could provide a framework for the convergence analysis sought in [5]. The main effect of de Roos's methods is to separate the age and time discretizations, while yielding an approximation that is dispersion-free in age, in order to provide a method that works in practice.

The main purpose of this paper is to provide a description and analysis of the use of higher order finite element spaces in the age variable. The time variable has been left continuous. The use of continuous time simplifies the presentation and analysis of the method as well as emphasizes the independence of the age discretization from any suitable time discretization. The methods are shown to be superconvergent in the age variable. We provide an example system that illustrates some of the benefits of using a higher order approximation space in age as well as highlights some of the interactions between the age and time discretizations in these methods.

2. A continuous model. We consider the age-dependent population model with nonlinear diffusion,

$$\partial_t u + \partial_a u = \nabla \cdot \big(k(x,p)\nabla u\big) - \mu(x,a,p)u, \qquad x \in \Omega,\ a > 0,\ t > 0, \tag{2.1}$$

where ∇ and $\nabla\cdot$ denote the gradient and the divergence, respectively, in x. The function $u(x,a,t)$ represents the distribution of individuals, $\Omega \subset \mathbb{R}^n$ represents the spatial domain, a represents age, and t represents time. The function $\mu > 0$ is the death rate. The total population density, p, is given by

$$p(x,t) = \int_0^\infty u(x,a,t)\,da, \qquad x \in \Omega,\ t > 0. \tag{2.2}$$

We have a birth condition

$$u(x,0,t) = b(x,u(x,\cdot,t)), \qquad x \in \Omega,\ t > 0, \tag{2.3}$$

that is dependent on the entire population distribution. We note that b is an operator whose second argument is a function defined on $\mathbb{R}^+$, where $\mathbb{R}^+$ denotes the nonnegative real numbers. The diffusion arises from the symmetric random motion of each individual (Fickian diffusion). We have a Neumann boundary condition, with ν denoting the outward normal to $\partial\Omega$,

$$k(x,p)\nabla u \cdot \nu = 0, \qquad x \in \partial\Omega,\ a > 0,\ t > 0, \tag{2.4}$$

that represents an isolated habitat. The initial condition is

$$u(x,a,0) = u_0(x,a), \qquad x \in \Omega,\ a > 0. \tag{2.5}$$

Langlais [11] proved the existence of unique nonnegative solutions for the case when k, μ, and β in (2.6) are independent of x, and Ω is bounded. A corresponding treatment for the system (2.1)–(2.5) is beyond the scope of this paper; we will concentrate on the numerical aspects of the problem. Thus we assume existence and uniqueness of smooth, nonnegative solutions.

We make several assumptions.

CONDITION 2.1. *There exists constants C_0 and C_1 such that, for $(x, p) \in \Omega \times \mathbb{R}$, k satisfies $0 < C_0 \leq k(x, p) \leq C_1$ and μ satisfies $0 < C_0 \leq \mu(x, a, p) \leq C_1$ for all a.*

CONDITION 2.2. *The functions $k(x, p)$ and $\mu(x, a, p)$ are uniformly Lipschitz continuous with respect to p with Lipschitz constants K_k and K_μ, respectively. The derivative $\partial_p k(x, p)$ exists. The derivative $\partial_a \mu(x, a, p)$ exists, is uniformly bounded by C_1 as a function of all its arguments, and $\|\partial_a \mu(x, \cdot, p)\|_{L^2(\mathbb{R}^+)} \leq C_1$ uniformly as a function of x and p.*

CONDITION 2.3. *The birth operator, $b : \Omega \times \left(L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)\right) \to \mathbb{R}^+$, is of the form*

$$b(x, \varphi(x, \cdot, t)) = \int_0^\infty \beta(x, a, \Phi)\varphi(x, a, t)\ da, \tag{2.6}$$

where $\beta \geq 0$ is the birth rate and Φ is the total population density, i.e., the integral of φ with respect to age. The function β is assumed to be uniformly Lipschitz continuous as a function of Φ. As a function of a, $\beta(x, a, \Phi)$ is in $H^1(\mathbb{R}^+)$, with its H^1-norm bounded independently of x and Φ; and it is also assumed that there is a positive a_{small} and a natural number k_0 such that $\beta(x, \cdot, \Phi)$ is a polynomial of degree at most k_0 on $(0, a_{small})$ with the coefficients bounded independently of x and Φ.

CONDITION 2.4. *The initial condition, $u_0(x, a)$, is bounded and nonnegative, and there exists $\tilde{a}_{max}$ such that $u_0(x, a) = 0$ for $a > \tilde{a}_{max}$.*

We note that the birth operator, b, is uniformly bounded and satisfies the Lipschitz condition

$$\begin{aligned} &|b(x, \varphi(x, \cdot, t)) - b(x, \psi(x, \cdot, t))| \\ &\qquad \leq K_b \left(\left(1 + \|\varphi\|_{L^1(\mathbb{R}^+)}\right) \left| \int_0^\infty (\varphi - \psi)\ da \right| + \|\varphi - \psi\|_{H^{-1}(\mathbb{R}^+)} \right), \end{aligned}$$

where $H^{-1}(\mathbb{R}^+)$ is the dual to $H^1(\mathbb{R}^+)$. The condition that β be polynomial in a for small a allows us to avoid some issues associated with the introduction of a new age interval.

Condition 2.4 is technically convenient and seems mild in light of the exponential decay of u in age [2]. A consequence of this condition is that $u(x, a, t)$ is zero if $a > \tilde{a}_{\max} + t$. Since we are dealing with time in a bounded interval in this work, the fact that the age is bounded above means that the behavior of β for very large a is unimportant.

3. An age and space discrete method. Let $D = \partial_t + \partial_a$. We reuse the symbol k to denote the form

$$k(\Phi; \varphi, v) = \int_\Omega k(x, \Phi)\nabla\varphi \cdot \nabla v\ dx;$$

the distinction between the form and $k(x,\Phi)$ should be clear from context. In variational form, for every $t \in \mathbb{R}^+$ and every $v \in H^1(\Omega)\otimes L^2(\mathbb{R}^+)$, we have

$$\int_0^\infty (Du,v)+k(p;u,v)+(\mu u,v)\ da = (b(x,u(x,\cdot,t)) - u(x,0,t), v(x,0)), \tag{3.1}$$

where $(\cdot,\cdot)$ denotes the L^2-inner product over Ω. Note that no regularity in a is required on v because both sides are zero independent of the choice of v, but we find it useful to think of $v(x,0)$ being the trace of v at $a=0$ when v is smooth.

Let $\mathcal{M}$ denote a finite dimensional subspace of $H^1(\Omega)$. Let $\{a_i\}_{i=0}^{-\infty}$ be a sequence such that $a_0 = \tilde{a}_{\max}$, $0 < a_{i+1} - a_i < \Delta a$, and $a_i \to -\infty$ as $i \to -\infty$. Let $\mathcal{J}$ be the set of a_i's and let $\check{\mathcal{J}}$ denote the set of $-a_i$'s. For a fixed nonnegative integer q, let $\mathcal{C}$ denote the space of all piecewise continuous functions over the partition of $\mathbb{R}$ defined by $\mathcal{J}$ such that $\varphi \in \mathcal{C}$ has the property that φ restricted to (a_i, a_{i+1}) is a polynomial of degree at most q for $i<0$ and φ is zero on (a_0,∞). We define a finite dimensional space in age that moves along the characteristic curves, $da/dt = 1$:

$$\mathcal{A}(t) = \left\{\varphi \in L^2(\mathbb{R}^+) : \varphi(\cdot) = \psi(\cdot - t)\big|_{\mathbb{R}^+},\ \psi \in \mathcal{C}\right\}.$$

Note that the dimension of $\mathcal{A}$ increases by $q+1$ as t goes from $-a_i - 0$ to $-a_i + 0$. This discretization will allow the numerical method to be free of numerical dispersion in age. We take $U(\cdot,\cdot,t) \in \mathcal{M}\otimes\mathcal{A}(t)$. For $t \notin \check{\mathcal{J}}$,

$$\begin{aligned}\int_0^\infty (DU,v)+k(P;U,v)+(\mu(x,a,P)U,v)\ da \\ = (b(x,U(x,\cdot,t)) - U(x,0,t), v(x,0,t))\end{aligned} \tag{3.2}$$

for every $v(\cdot,\cdot,t) \in \mathcal{M}\otimes\mathcal{A}(t)$. So that U is defined across points in $\mathcal{J}$, we require U to be a continuous mapping of time into $L^2(\Omega)\otimes L^2(\mathbb{R}^+)$. The total population density is approximated by

$$P(x,t) = \int_0^\infty U(x,a,t)\ da.$$

We want to emphasize that the function U is differentiable in the characteristic direction so that DU makes sense. Functions in $\mathcal{A}(t)$ are discontinuous, but those discontinuities move along characteristic directions.

The system defined by (3.2) is just a set of ordinary differential equations when $t \notin \check{\mathcal{J}}$. Consider $t \in \check{\mathcal{J}}$. Since the part that is nonstandard is related to the age variables, we will suppress the x variables. Let $\{\varphi_i\}$ denote a basis for $\mathcal{C}$, where each φ_i has support in one interval $[a_k, a_{k+1}]$. Then a natural basis for $\mathcal{A}(t)$ is of the form $\{\varphi_i(\cdot - t)\}$; the functions in the basis for $\mathcal{A}$ are restricted to $\mathbb{R}^+$, and there are only a finite number of these that are nontrivial. The function U is expressed as

$$U(a,t) = \sum_i c_i(t)\varphi_i(a-t) \text{ so that } DU = \sum_i c_i'(t)\varphi_i(a-t).$$

While it is natural to look at a set of relations of the form

$$\int_0^\infty DU(a,t)\varphi_j(a-t)da = F(t,U,\varphi_j) + (b(U) - U(0,t))\varphi_j(0-t) \tag{3.3}$$

as a set of ordinary differential equations for the c_i's, there is a difficulty because the coefficient of the vector of c_i''s is singular at a transition. Note that

$$\int_0^\infty DU\varphi_j(a-t)da + U(0,t)\varphi_j(0-t) = \frac{d}{dt}\int_0^\infty U\varphi_j(a-t)da.$$

Thus (3.3) can be written as $m_j' = F(t, U, \varphi_j) + b(U)\varphi_j(0-t)$, where

$$m_j(t) = \int_0^\infty U(a,t)\varphi_j(a-t)da;$$

these are the natural variables. The initial values for the new m_j's added to the system when t crosses a point of $\check{\mathcal{J}}$ are clearly zero because of the assumed continuity of U (as a map into $L^2(\mathbb{R}^+)$) at such points. The m_j's already present are continuous at these transitions. We must check whether the birth operator is Lipschitz with respect to these natural variables. This is easy to confirm in the case in which the function β is polynomial of degree at most k_0 near $a = 0$, and that is why we chose to address birth operators of that form. What is needed is that we can choose a basis such that the coefficients of the L^2-projection of $\beta(a)$ into $\mathcal{A}$ are bounded. This is trivial away from $a = 0$ because of the equivalence of norms on finite dimensional spaces.

4. Error analysis. Wheeler, in her analysis for parabolic equations [15], showed the value of choosing the right projection in constructing an argument; in her case it was the elliptic projection. In this paper we use a tensor product projection based on an elliptic projection in space and an L^2-projection in age.

It is convenient to use $H^{-1}(\Omega)$ as the dual to $H^1(\Omega)$. Let $\|\cdot\|$, $\|\cdot\|_{L^\infty}$, $\|\cdot\|_{H^1}$, and $\|\cdot\|_{H^{-1}}$ denote the L^2, L^∞, H^1, and H^{-1} norms over Ω, respectively. Suppose that Υ is a normed space with norm $\|\cdot\|_\Upsilon$. Then, for any sufficiently nice function $\varphi : \mathbb{R}^+ \to \Upsilon$, let

$$|\!|\!|\varphi|\!|\!|_\Upsilon^2 = \int_0^\infty \|\varphi(a)\|_\Upsilon^2 \; da.$$

A lack of a subscript indicates $\Upsilon = L^2(\Omega)$. For $\varphi : \Omega \to \Upsilon$, define

$$\|\varphi\|_{L^p(\Omega,\Upsilon)} = \Big\| \, \|\varphi(x)\|_\Upsilon \, \Big\|_{L^p(\Omega)}.$$

We show that the approximate solution U is close to a function X, which is the elliptic projection in space and the L^2-projection in age of the true solution u. Let $\mathsf{A}(t) : L^2(\mathbb{R}^+) \to \mathcal{A}(t)$ denote the L^2-projection. To construct X we first project into space. For each (a,t), we take $\tilde{X}(a,t) \in \mathcal{M}$ such that $k(p; u - \tilde{X}, v) = 0$ for all $v \in \mathcal{M}$ and such that

$$\int_\Omega |u - \tilde{X}| \; dx = 0.$$

Similarly, for each t, we take $Y(t) \in \mathcal{M}$ to satisfy $k(p; p - Y, v) = 0$ for all $v \in \mathcal{M}$ and

$$\int_\Omega |p - Y| \; dx = 0.$$

To project into age, we choose $X(t) \in \mathcal{M} \otimes \mathcal{A}(t)$ such that $X(t) = \mathsf{A}(\tilde{X}(a,t))$. We set

$$\vartheta = U - X, \quad \eta = u - \tilde{X}, \quad \tilde{\eta} = \tilde{X} - X, \quad \varpi = P - Y, \text{ and } \sigma = p - Y.$$

In the following estimate we will suppose that ∇Y and the L^2-norm in age of ∇X are uniformly bounded. We could instead add conditions on $\mathcal{M}$, Ω, and u that would imply these bounds, but this would add complexity with no benefit in understanding why the numerics work. We add the following condition.

CONDITION 4.1. *We suppose the quantities*

$$\|\!|u|\!\|_{L^\infty}, \quad \|u\|_{L^\infty(\Omega\times\mathbb{R}^+)}, \quad \|u\|_{L^\infty(\Omega,L^1(\mathbb{R}^+))}, \quad K_k\|\!|\nabla X|\!\|_{L^\infty}, \|p\|_{L^\infty}, \text{ and } K_k\|\nabla Y\|_{L^\infty}$$

are bounded uniformly in time.

THEOREM 4.1. *Let*

$$\theta_1(t) = \int_0^t C_0(\|\!|\vartheta|\!\|_{H^1}^2 + \|\varpi\|_{H^1}^2)(\tau)\ d\tau,$$

$$\epsilon(t) = \int_0^t (\|\!|\eta|\!\|^2 + \|\sigma\|^2 + \|\!|D\eta|\!\|_{H^{-1}}^2 + \|\partial_t\sigma\|_{H^{-1}}^2 + (\Delta a)^2\|\!|\tilde{\eta}|\!\|^2 + \|\eta(x,0)\|^2)(\tau)\ d\tau.$$

Assume Conditions 2.1, 2.2, 2.3, 2.4, and 4.1 *hold. There exists $C^*(t) > 0$ (dependent only on K_b, K_μ, C_0, C_1, and the bounds in Condition* 4.1*) such that*

$$(\|\!|\vartheta|\!\|^2 + \|\varpi\|^2 + \theta_1)(t) \le C^*(t)\big((\|\!|\vartheta|\!\|^2 + \|\varpi\|^2)(0) + \epsilon(t)\big).$$

Remark. This result shows superconvergence of one additional power of Δa in the age variable since only $\tilde{\eta}$ involves approximation in age. Hence, as a function of age, U is closer to the L^2-projection in age of u than it is to u itself, at least for Δa sufficiently small.

Proof. For this proof C will denote an arbitrary constant with dependencies not greater than those of C^*. When only a single argument is given to U, u, η, $\tilde{\eta}$, or ϑ, that argument denotes age.

Subtract (3.1) from (3.2) and let $v = \vartheta$ to get

$$\begin{aligned}(4.1)\quad \int_0^\infty (D(\vartheta - \eta - \tilde{\eta}), \vartheta) + k(P; U, \vartheta) - k(p; u, \vartheta) + (\mu(P)U, \vartheta) - (\mu(p)u, \vartheta)\ da \\ + (U(0) - u(0), \vartheta(0)) = (b(U) - b(u), \vartheta(0)).\end{aligned}$$

By orthogonality, for $v(\cdot,\cdot,t) \in \mathcal{M} \otimes \mathcal{A}(t)$,

$$\int_0^\infty (\tilde{\eta}, v)\ da = 0. \tag{4.2}$$

For $t \notin \check{\mathcal{J}}$, let $\delta > 0$ be such that $(t-\delta, t+\delta)\cap\check{\mathcal{J}} = \emptyset$. For a given $v(\cdot,\cdot,t) \in \mathcal{M}\otimes\mathcal{A}(t)$ and $-s \in (t-\delta, t+\delta)$, take $v(\cdot,\cdot,s) \in \mathcal{M}\otimes\mathcal{A}(s)$ such that v is constant along characteristics. By (4.2) we have, for $0 < \Delta t < \delta$,

$$\begin{aligned}0 &= \frac{1}{\Delta t}\int_0^\infty \big(\tilde{\eta}(\cdot,a,t+\Delta t), v(\cdot,a,t+\Delta t)\big) - \big(\tilde{\eta}(\cdot,a,t), v(\cdot,a,t)\big)\ da \\ &= \frac{1}{\Delta t}\int_0^\infty \big(\tilde{\eta}(\cdot,a+\Delta t,t+\Delta t) - \tilde{\eta}(\cdot,a,t), v(\cdot,a+\Delta t,t+\Delta t)\big)\ da \\ &\quad + \frac{1}{\Delta t}\int_0^{\Delta t} \big(\tilde{\eta}(\cdot,a,t+\Delta t), v(\cdot,a,t+\Delta t)\big)\ da \\ &\quad + \frac{1}{\Delta t}\int_0^\infty \big(\tilde{\eta}(\cdot,a,t), v(\cdot,a+\Delta t,t+\Delta t) - v(\cdot,a,t)\big)\ da.\end{aligned}$$

In this expression the last term is zero because v is constant along characteristics. Taking limits we see that for $v(\cdot,\cdot,t) \in \mathcal{M} \otimes \mathcal{A}(t)$,

$$\int_0^\infty \big(D\tilde{\eta}(\cdot,a,t), v(\cdot,a,t)\big)\ da + \big(\tilde{\eta}(\cdot,0,t), v(\cdot,0,t)\big) = 0.$$

Hence, with $v = \vartheta$, we note that

$$\begin{aligned} (4.3) \qquad \big(U-u,\vartheta\big)(0) - \int_0^\infty \big(D\tilde{\eta},\vartheta\big)\ da &= \big(\vartheta-\eta-\tilde{\eta},\vartheta\big)(0) - \int_0^\infty \big(D\tilde{\eta},\vartheta\big)\ da \\ &= \|\vartheta(0)\|^2 - \big(\eta,\vartheta\big)(0) \\ &\geq \frac{3}{4}\|\vartheta(0)\|^2 - \|\eta(0)\|^2. \end{aligned}$$

Rearranging terms in (4.1) and applying (4.2) and (4.3) gives

$$\begin{aligned} &\int_0^\infty (D\vartheta,\vartheta) + k(P;\vartheta,\vartheta) + (\mu(P)\vartheta,\vartheta)\ da + \frac{3}{4}\|\vartheta(0)\|^2 \\ &\qquad \leq \int_0^\infty (k(x,Y)-k(x,P), \nabla X\cdot\nabla\vartheta) + (k(x,p)-k(x,Y), \nabla X\cdot\nabla\vartheta)\ da \\ &\qquad\quad + \int_0^\infty ((\mu(p)-\mu(P))u,\vartheta) + (D\eta,\vartheta) + (\mu(P)(\eta+\tilde{\eta}),\vartheta)\ da \\ &\qquad\quad + \ (b(U)-b(u),\vartheta(0)) + \|\eta(0)\|^2. \end{aligned}$$

We have the equality

$$\int_0^\infty (D\vartheta,\vartheta)\ da = \frac{1}{2}\int_0^\infty D\|\vartheta\|^2\ da = \frac{1}{2}\partial_t\|\vartheta\|^2 - \frac{1}{2}\|\vartheta(0)\|^2.$$

Using Conditions 2.1–2.2, Hölder's inequality, and the arithmetic-geometric inequality, $yz \leq \frac{1}{2}(\varepsilon y^2 + (1/\varepsilon)z^2)$, we get the following bounds:

$$\begin{aligned} \int_0^\infty k(P;\vartheta,\vartheta) + (\mu(P)\vartheta,\vartheta)\ da &\geq C_0\|\vartheta\|_{H^1}^2, \\ \int_0^\infty (k(x,Y)-k(x,P),\nabla X\cdot\nabla\vartheta)\ da &\leq \int_0^\infty (K_k|\varpi|, |\nabla X\cdot\nabla\vartheta|)\ da \\ &\leq \frac{2K_k^2}{C_0}\|\nabla X\|_{L^\infty}^2\|\varpi\|^2 + \frac{C_0}{8}\|\vartheta\|_{H^1}^2, \\ \int_0^\infty (k(x,Y)-k(x,P),\nabla X\cdot\nabla\vartheta)\ da &\leq \frac{2K_k^2}{C_0}\|\nabla X\|_{L^\infty}^2\|\sigma\|^2 + \frac{C_0}{8}\|\vartheta\|_{H^1}^2, \\ \int_0^\infty ((\mu(p)-\mu(P))u,\vartheta)\ da &\leq \int_0^\infty (K_\mu|P-p|, |u\vartheta|)\ da \\ &\leq \frac{K_\mu}{2}\|u\|_{L^\infty}\left(2(\|\varpi\|^2+\|\sigma\|^2) + \|\vartheta\|^2\right), \\ \int_0^\infty (D\eta,\vartheta)\ da &\leq \frac{1}{C_0}\|D\eta\|_{H^{-1}}^2 + \frac{C_0}{4}\|\vartheta\|_{H^1}^2, \\ \int_0^\infty (\mu(P)\eta,\vartheta)\ da &\leq \frac{C_1}{2}(\|\eta\|^2 + \|\vartheta\|^2). \end{aligned}$$

Let $\bar{\mu}$ denote the average of μ in age over each interval of the age discretization. Then

$$\begin{aligned}\int_0^\infty (\mu(P)\tilde{\eta}, \vartheta)\; da &= \int_0^\infty \big((\mu(P) - \bar{\mu}(P))\tilde{\eta}, \vartheta\big)\; da \\ &\le \frac{1}{2}\big(\|\mu(P) - \bar{\mu}(P)\|^2_{L^\infty(\Omega\times\mathbb{R}^+)} |||\tilde{\eta}|||^2 + |||\vartheta|||^2\big) \\ &\le \frac{(\Delta a)^2}{8} C_1^2 |||\tilde{\eta}|||^2 + \frac{1}{2}|||\vartheta|||^2.\end{aligned}$$

For the birth term we make the bound

$$\begin{aligned}(b(U) - b(u), \vartheta(0)) &\le \|b(U) - b(u)\|^2 + \frac{1}{4}\|\vartheta(0)\|^2 \\ &\le 3K_b^2\Big((1 + \|u\|^2_{L^\infty(\Omega, L^1(\mathbb{R}^+))})\|P - p\|^2 \\ &\qquad + \|U - u\|^2_{L^2(\Omega, H^{-1}(\mathbb{R}^+))}\Big) + \frac{1}{4}\|\vartheta(0)\|^2.\end{aligned}$$

We combine the above inequalities and use the fact that $\|\tilde{\eta}\|_{L^2(\Omega, H^{-1}(\mathbb{R}^+))} \le \frac{\Delta a}{\pi}|||\tilde{\eta}|||$ (see Appendix A of [2]) to get

$$\begin{aligned}\partial_t |||\vartheta|||^2 + C_0 |||\vartheta|||^2_{H^1} \le C\Big(&\|\varpi\|^2 + |||\vartheta|||^2 + \|\sigma\|^2 + |||\eta|||^2 + |||D\eta|||^2_{H^{-1}} \\ &+ (\Delta a)^2 |||\tilde{\eta}|||^2 + \|\eta(0)\|^2\Big).\end{aligned} \tag{4.4}$$

Before we can use the above evolution inequality to get bounds on the error, we need corresponding relationships for the total population density. We integrate (2.1) over a and take the inner product with a test function $v \in \mathcal{M}$ to obtain

$$(\partial_t p, v) + k(p; p, v) + \left(\int_0^\infty \mu(p) u\; da\,, v\right) = (b(u), v). \tag{4.5}$$

For the approximate total population density we have

$$(\partial_t P, v) + k(P; P, v) + \left(\int_0^\infty \mu(P) U\; da\,, v\right) = (b(U), v). \tag{4.6}$$

We subtract (4.5) from (4.6) and let $v = \varpi$ to get

$$\begin{aligned}\frac{1}{2}\partial_t\|\varpi\|^2 + k(P; P, \varpi) - k(p; p, \varpi) + \left(\int_0^\infty \mu(P)U - \mu(p)u\; da, \varpi\right) \\ = (b(U) - b(u), \varpi) + (\partial_t \sigma, \varpi).\end{aligned}$$

This has a form similar to (4.1). We have the bound

$$\begin{aligned}\left(\int_0^\infty \mu(P)\tilde{\eta}\; da, \varpi\right) &= \left(\int_0^\infty (\mu(P) - \bar{\mu}(P))\tilde{\eta}\; da, \varpi\right) \\ &\le \left(\|\mu(x,\cdot,P) - \bar{\mu}(x,\cdot,P)\|^2_{L^2(\mathbb{R}^+)}\|\tilde{\eta}(x,\cdot,t)\|^2_{L^2(\mathbb{R}^+)}, \varpi\right) \\ &\le \frac{(\Delta a)^2 C_1^2}{2\pi^2}|||\tilde{\eta}|||^2 + \frac{1}{2}\|\varpi\|^2.\end{aligned}$$

Using bounds similar to those for (4.1) for the other terms gives

$$\partial_t \|\varpi\|^2 + C_0 \|\varpi\|_{H^1}^2 \le C\Big(\|\varpi\|^2 + \|\!|\vartheta|\!\|^2 + \|\sigma\|^2 + \|\!|\eta|\!\|^2 + \|\partial_t \sigma\|_{H^{-1}}^2 + (\Delta a)^2 \|\!|\tilde{\eta}|\!\|^2 \Big). \tag{4.7}$$

By adding (4.4) and (4.7) and applying a Gronwall's lemma,[1] we obtain the stated result. □

5. Computational example. In this section we provide a computational example that shows some of the benefits of using a higher order approximation space in the age variable. In particular, we are able to use a coarser age discretization for the same or better error.

The example system in [2] illustrates the importance of being able to decouple the age and time discretizations so that the age and time steps are neither uniform nor equal. The spatial dynamics of the problem require small time steps for accurate resolution. The small time steps taken in the simulation, particularly the initial step, are caused by roughness in space. The behavior in age is relatively smooth, which in turn calls for a much coarser discretization in age than in time.

The example presented in [2] was meant to illustrate the need for a method that discretizes age and time separately because of the influence of space. It does not clearly illustrate two aspects of the interaction of age and time. First, it is not clear what is needed to align the introduction of an age interval with the start of a time step. Second, it does not illustrate the level to which the age dynamics of a system will determine the size of the time step.

We present an example system that illustrates the benefits of using higher order approximation spaces in age, as well as some aspects of the interaction of age and time in these methods. In order to achieve the latter goal, we assume uniformity in space. Because the dynamics in age can be, and often are, independent of space, the benefits of using higher order polynomial spaces in age will generalize to systems that include spatial dynamics.

We consider the system (2.1)–(2.5) with $k = 0$. We use the birth term,

$$b(x, u(x,\cdot,t)) = \int_0^\infty 5a\, u \; da,$$

so that fecundity increases linearly with age. For the death modulus, we use

$$\mu(x,a,p) = \mu(a) = \frac{10e^{10(a-0.8)}}{e^{10(a-0.8)} + e^{-10(a-0.8)}} + \frac{1}{2}.$$

This represents a situation where mortality remains low until around a certain age, at which point it increases dramatically. This is the case in *Proteus mirabilis* swarm colony development [6].

For the initial condition, we use a population of older organisms,

$$u_0(x,a) = 128|a-0.5|^3 - 48(a-0.5)^2 + 1,$$

if $|a - 0.5| < 0.25$, and $u_0(x,a) = 0$, otherwise.

[1] Assume $u, b, c \ge 0$ are continuous and $g \ge 0$ is differentiable. Then $g'(t) + b(t) \le c(t) + u(t)g(t)$ implies $g(t) + \int_0^t b(\tau)\; d\tau \le \exp(\int_0^t u(\tau)\; d\tau)(g(0) + \int_0^t c(\tau)\; d\tau)$.

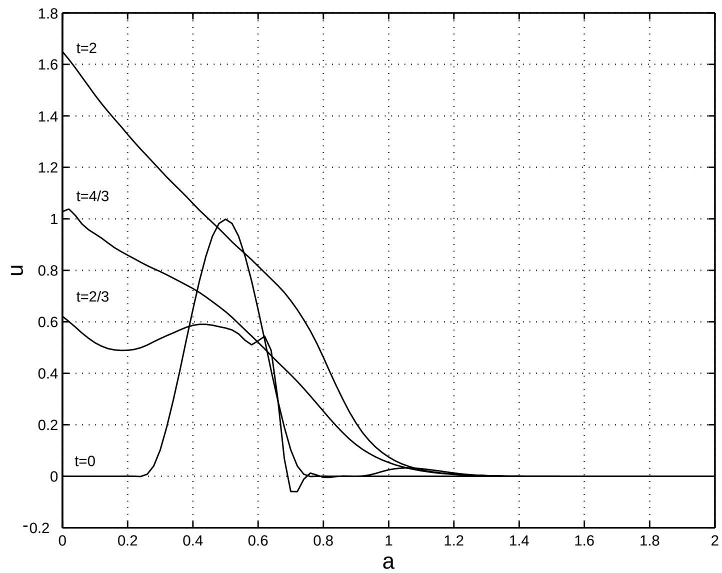

FIG. 5.1. *Profiles of the population density, u. The profiles are $t = 2/3$ apart.*

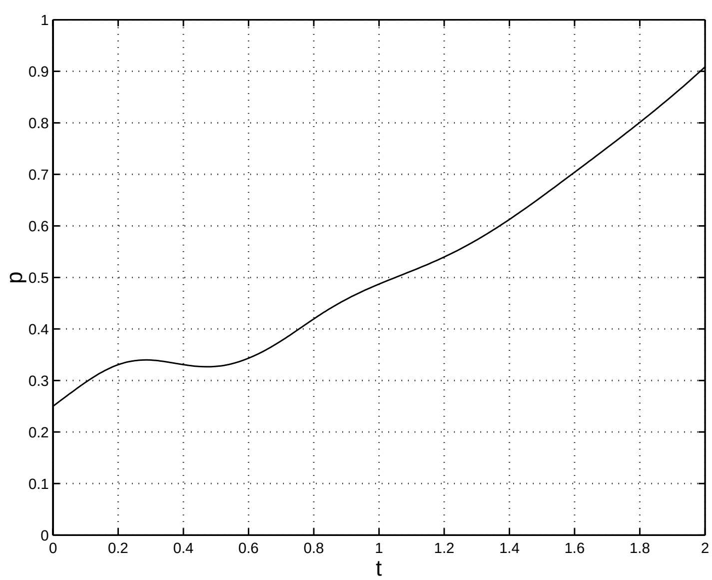

FIG. 5.2. *The total population density, p.*

We take the temporal domain to be $[0, 2]$. We find that truncating the age domain to $[0, 2]$ is sufficient. We assume uniformity of the solution over the spatial domain Ω.

We implement step-size control in time via step-doubling (without extrapolation) [1, 7]. This means that for each time step we take a step of size Δt and compare it to the solution obtained by taking two steps of size $\Delta t/2$. We adjust a parameter that limits local truncation error so that the simulation is well resolved in time.

For the age discretization, we assume $\mathcal{J}$ is uniform with age intervals of size Δa.

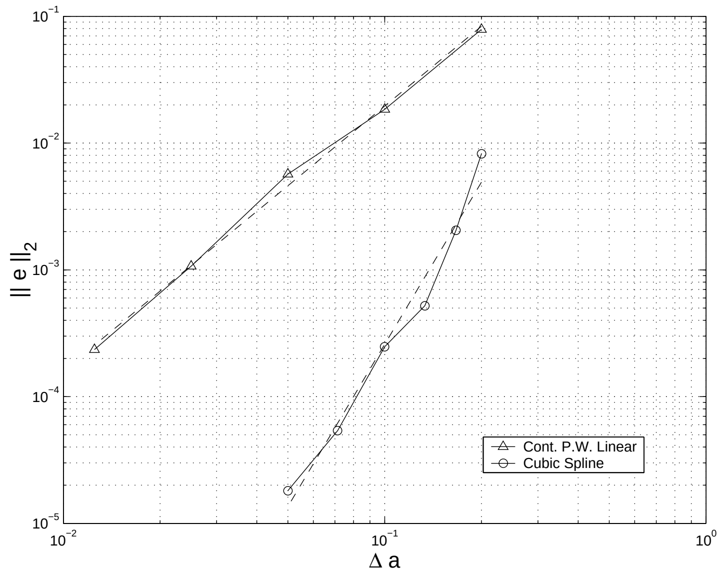

FIG. 5.3. *Convergence study for $q = 0$ and $q = 1$ showing second order and fourth order convergence of the method, respectively. The comparison is of the computed solution, u, at time $t = 2$. The slope of the least squares fit for the piecewise constant case postprocessed to continuous piecewise linear functions is approximately* 2.09*. The slope of the least squares fit for the discontinuous piecewise linear case postprocessed to cubic splines is approximately* 4.25.

In other words, all age intervals that are not the birth interval are of length Δa. We study the convergence of the method with piecewise constants postprocessed to continuous piecewise linear functions ($q = 0$) and with discontinuous piecewise linear functions postprocessed to cubic splines ($q = 1$). The postprocessing for piecewise constants was discussed in [2]; it involves using knot values that are obtained from a line connecting the midpoints of adjacent intervals. The cubic splines can be produced from the discontinuous linear functions in several ways; here we used the two Gauss points in each of two adjacent intervals to define a cubic that is used to give knot values and slopes. This is a natural choice since the L^2-projection into discontinuous piecewise linear functions is superconvergent at the two Gauss points.

Figure 5.1 shows solution profiles $t = 2/3$ apart for the simulation using $q = 1$ and $\Delta a = 0.1$. The solution at $t = 2/3$ has a discontinuity because the initial condition does not contain any newborns. However, this discontinuity dies out over time. Figure 5.2 shows the growth of the total population density. There is a period of population decline, due to the die-off of the initial population, before the population enters a stage of exponential growth.

Figure 5.3 shows the results of a convergence study using $q = 0$ and $q = 1$ with postprocessing. The error is determined by comparison with the numerical solution solved with $\Delta a = 6.25 \times 10^{-3}$ for the case of $q = 0$ and $\Delta a = 2.5 \times 10^{-2}$ for $q = 1$. We get the expected result that the use of discontinuous piecewise linear functions gives fourth order convergence with much better initial error than the second order convergence given by the use of piecewise constants.

Figure 5.4 shows the time steps taken for the simulation using $q = 1$ and $\Delta a = 0.1$. We find that the time step needed to resolve this simulation is roughly 10^{-2}, with the exception of a trough at $t \approx 1.8$ that corresponds to the die-off of the relatively

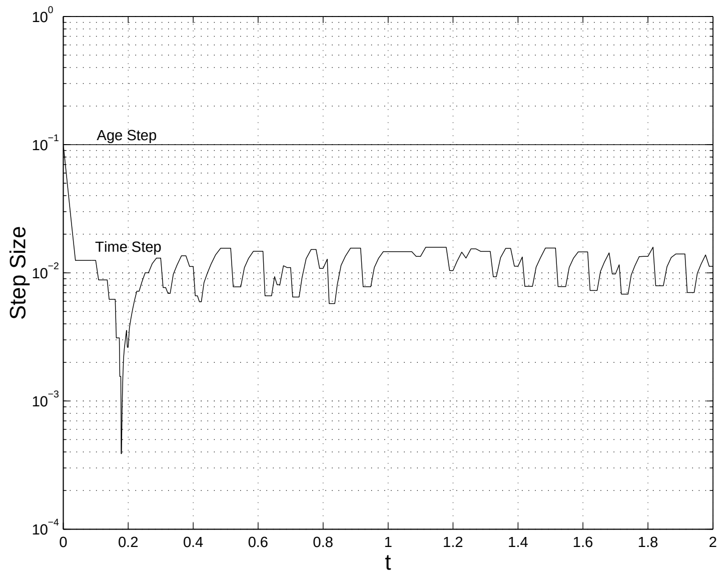

FIG. 5.4. *Step sizes for the age and time discretizations. There were* 204 *accepted steps and* 22 *rejected steps during the simulation. The smallest step size was approximately* 3.88×10^{-4} *taken at* $t \approx 1.8$*. This trough corresponds to the die-off of the relatively large initial population of older individuals.*

large initial population of older individuals. The need for this small time step is due to the increased complexity of the underlying problem at this point, not the moving age grid.

The restrictions on the time steps imposed by the age discretization are due to the need to introduce a new age interval at the start of a time step. This requires that $\Delta t \leq \Delta a$. Moreover, this restriction may require a slightly smaller time step before the introduction of a new age interval at the birth boundary. Smaller time steps may also be needed during the initial birthing into a new age interval. These cause the minor time step fluctuations we see throughout the latter part of the simulation.

REFERENCES

[1] B. P. AYATI, *Methods for Computational Population Dynamics*, Ph.D. thesis, University of Chicago, Chicago, IL, 1998.

[2] B. P. AYATI, *A variable time step method for an age-dependent population model with nonlinear diffusion*, SIAM J. Numer. Anal., 37 (2000), pp. 1571–1589.

[3] A. M. DE ROOS, *Numerical methods for structured population models: The escalator boxcar train*, Numer. Methods Partial Differential Equations, 4 (1989), pp. 173–195.

[4] A. M. DE ROOS, *A gentle introduction to physiologically structured population models*, in Structured-population Models in Marine, Terrestrial, and Freshwater Systems, Population and Community Biology Series 18, S. Tuljapurkar and H. Caswell, eds., Chapman & Hall, New York, 1997, pp. 119–204.

[5] A. M. DE ROOS AND J. A. J. METZ, *Towards a numerical analysis of the escalator boxcar train*, in Differential Equations with Applications in Biology, Physics and Engineering, Lecture Notes in Pure and Appl. Math. 133, J. A. Goldstein, F. Kappel, and W. Schappacher, eds., Marcel Dekker, New York, 1991, pp. 91–113.

[6] S. E. ESIPOV AND J. A. SHAPIRO, *Kinetic model of* Proteus mirabilis *swarm colony development*, J. Math. Biol., 36 (1998), pp. 249–268.

[7] C. W. GEAR, *Numerical Initial Value Problems in Ordinary Differential Equations*, Prentice–

Hall, Englewood Cliffs, NJ, 1971.

[8] M.-Y. Kim, *Galerkin methods for a model of population dynamics with nonlinear diffusion*, Numer. Methods Partial Differential Equations, 12 (1996), pp. 59–73.

[9] M.-Y. Kim and E.-J. Park, *Mixed approximation of a population diffusion equation*, Comput. Math. Appl., 30 (1995), pp. 23–33.

[10] B. W. Kooi and S. A. L. M. Kooijman, *Discrete event versus continuous approach to reproduction in structured population dynamics*, Theoret. Population Biol., 56 (1999), pp. 91–105.

[11] M. Langlais, *A nonlinear problem in age-dependent population diffusion*, SIAM J. Math. Anal., 16 (1985), pp. 510–529.

[12] L. Lopez and D. Trigiante, *A finite difference scheme for a stiff problem arising in the numerical solution of a population dynamic model with spatial diffusion*, Nonlinear Anal., 9 (1985), pp. 1–12.

[13] M. Pascual and S. A. Levin, *Spatial scaling in a benthic population model with density-dependent disturbance*, Theoret. Population Biol., 56 (1999), pp. 106–122.

[14] O. Rauprich, M. Matsushita, K. Weijer, F. Siegert, S. E. Esipov, and J. A. Shapiro, *Periodic phenomena in* Proteus mirabilis *swarm colony development*, J. Bacteriol., 178 (1996), pp. 6525–6538.

[15] M. F. Wheeler, *A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations*, SIAM J. Numer. Anal., 10 (1973), pp. 723–759.